Clustering

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Clustering refers to partitioning a set into "dissimilar" subsets of 'similar' elements.

Usually a well-defined objective.

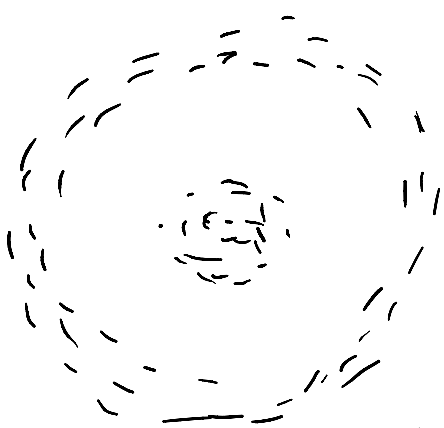
E.g. k-means, k-median, k-center, diameter.

— define

or with the goal of recovering some ground truth.

— all are NP-hard

No universal clustering criterion.



Depends on the context/application.

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set

K-center: – start with any point in given set as first center. $C = \{c_1\}$

Repeat $K-1$ times [– add farthest point to C

Thm. Greedy algorithm is a factor 2 approximation.

Pf. Suppose OPT is r.

claim: For the centers $c_1 \dots c_K$ found by GREEDY, max distance to nearest center $\le 2r$.

If not,

$\Rightarrow \exists\, K+1$ pts $c_1 \dots c_{K+1}$

s.t. $d(c_i, c_j) > 2r$.

$\Rightarrow$ No two of $\{c_1 \dots c_{K+1}\}$ can belong to same cluster of radius r.

The Spectral Approach

- Project to span of top k singular vectors of $A \in \mathbb{R}^{n \times d}$

- Cluster in $\mathbb{R}^k$.

between

Idea: this should shrink distance between x and nearest center.

$$n \overbrace{\begin{pmatrix} \\ & A & \\ \\ \end{pmatrix}}^{d} - \begin{pmatrix} — c_1 — \\ — c_2 — \\ C \\ \\ \end{pmatrix}$$

centers.

$\sigma^2(C) = \dfrac{\|A - C\|_2}{n}$: average variance of clusters.

Thm If $\|C_i - C_j\| > 15 \frac{k}{\varepsilon} \cdot \sigma(C)$ $\forall i \neq j$ and each cluster has $\leq \varepsilon n$, then Spectral Clustering finds C' that differs from C in at most $\varepsilon^2 \cdot n$ points.

Algo.
1. Project to top k right singular vectors of A.

Repeat k times. [2. Take a random row, include all points within distance $6 \frac{k \sigma(C)}{\varepsilon}$.

$\|D - A\|_2$

$$A_k = \operatorname*{argmin}_{D:\, rk(D) \le k} \|D - A\|_2$$

<u>Lemma</u>. for any C of rank k, $\|A_k - C\|_F^2 \le 8k \|A - C\|_2^2$ any A.

<u>Pf</u>. (*) $\|A_k - C\|_F^2 \le 2k \|A_k - C\|_2^2$

Since $A_k - C$ has rank $\le 2k$.

$$\|A_k - C\|_2 \le \|A_k - A\|_2 + \|A - C\|_2$$

(**) $$\le 2\|A - C\|_2$$

(*) + (**) $\Rightarrow$ $\|A_k - C\|_F^2 \le 8k \|A - C\|_2^2$.

Pf (of Thm). Let v_i be i^{th} row of A_k.

<u>Claim</u>. Most v_i are within distance $\frac{3k}{\varepsilon}\sigma(C)$ of their center.

Let $B = \{i : \|v_i - C_i\| > \frac{3k}{\varepsilon}\sigma(C)\}$.

Then $\|A_k - C\|_F^2 \ge |B| \cdot \frac{9k^2}{\varepsilon^2} \cdot \sigma(C)^2$

$\le 8k\,\sigma(C)^2 \cdot n$

$\Rightarrow \quad |B| < \frac{\varepsilon^2}{k} \cdot n.$

For $i, j \in$ same cluster and $\notin B$,

$$\|v_i - v_j\| \le \frac{6k}{\varepsilon}\sigma(C). \quad \checkmark$$

i, j different cluster, $\notin B$

$$\|v_i - v_j\| > \frac{15k}{\varepsilon}\sigma(C) - \frac{6k}{\varepsilon}\sigma(C) = \frac{9k}{\varepsilon}\sigma(C).$$

Hence if we pick point not in B as the seed, all k times, all points not in B will be correctly classified.

$$\Pr(\text{we pick point} \notin B) \ge \left(1 - \frac{\varepsilon^2}{k}\right) \cdot \left(1 - \frac{(\varepsilon - \varepsilon^2)}{k}\right)^{k-1}$$

$$\ge 1 - \frac{\varepsilon}{\cancel{k}} \cdot \cancel{k} = 1 - \varepsilon.$$

Example 1 Mixture of k Gaussians, each with max covariance σ^2.

Then $\sigma(C) \le c_1 \sigma$ (max distance to own center in one direction)

Separation needed between centers: $15\frac{k}{\varepsilon}\sigma(C) = O\left(\frac{k}{\varepsilon} \cdot \sigma\right)$.

(a bit worse than what we got)

Example 2 Stochastic Block Models or Planted Partitions.

or Planted Partitions.

$$\begin{array}{|c|c|c|}\hline p & q & q \\ \hline & p & \\ \hline & & p \\ \hline\end{array}$$

$$\Pr((i,j) \in E) = \begin{cases} p & i,j \in \text{same block} \\ q & \text{different blocks.} \end{cases}$$

A: adjacency matrix of G.

$$\mathbb{E}(A) = C = \begin{pmatrix} p\,p \cdots p & q \cdots\cdots q \\ & \vdots \\ & \vdots \end{pmatrix}$$

Problem: Recover "planted" partition.

Apply spectral algorithm.

$$\|\mu_i - \mu_j\|^2 = \cdot (p-q)^2 \cdot \frac{n}{k}$$

different clusters.

What about $\|A-C\|_2$?

$A-C$ is a random matrix with $\mathbb{E}(A-C)=0$ and independent entries.

Thm. R random with independent entries $\in [-1,1]$

$\mathbb{E}(R_{ij})=0$ $\mathbb{E}(R_{ij}^2) \le \sigma^2$. $\|R\| \le (2+o(1))\,\sigma\sqrt{n}$.

$\mathbb{E}(R_{ij})=0$, $\mathbb{E}(R_{ij}^2) \le \sigma^2$,

Then with prob. $1-o(1)$, $\|R\|_2 \le (2+o(1))\sigma\sqrt{n}$.

So, $\sigma(C)^2 = \dfrac{\|A-C\|_2^2}{n} = O(p)$

So by the spectral clustering theorem,

it suffices to have

$$\|\mu_i - \mu_j\|^2 \ge (p-q)^2 \cdot \frac{n}{k} > c\frac{k^2}{\varepsilon^2} \cdot p > \left(\frac{15k}{\varepsilon}\right)^2 \cdot \sigma(C)^2$$

or $|p-q| > c' \dfrac{k^{3/2}}{\varepsilon}\sqrt{\dfrac{p}{n}}$

if we set $p = \dfrac{a}{n}$, $q = \dfrac{b}{n}$

then this says. $|a-b| = \Omega(k^{3/2}) \cdot \sqrt{a}$

suffices.

This is information theoretically tight up to constant factors.

In fact for $k=2$, $(a-b)^2 > 2a$

is necessary and sufficient!

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random matrix bound?

Q. How to prove the random matrix bound?

Planted Clique: