ICA

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Model

Given samples $x = As$

unknown $A \in \mathbb{R}^{n \times d}$

$$s = \begin{pmatrix} s_1 \\ \vdots \\ s_d \end{pmatrix}$$ has independent coordinates

We can assume $\mathbb{E}(s) = 0$ (else recenter x)

and $\mathbb{E}(s_i^2) = 1$

( ) ( ) since scaling s_i is same as scaling $A^{(i)}$.

<u>Problem</u> Recover colums of A up to sign.

Q. Is this uniquely identifiable?

Suppose $s_1, s_2 \sim N(0, 1)$

then for any rotation A, $x = As \sim N(0, I_2)$

$\boxed{\text{\underline{Thm}. If at most one component of } s \text{ is Gaussian, then } A \text{ is uniquely identifiable}}$

Gaussian, then A is uniquely ...
up to signs of its columns.

How?
Let's try moments.

$$\mathbb{E}(x) = \mathbb{E}(As) = A\,\mathbb{E}(s) = 0$$

$$\mathbb{E}(xx^T) = A\,\mathbb{E}(ss^T)A^T = AA^T$$

So we can again apply a linear transformation to make $\mathbb{E}(xx^T) = I = AA^T$ if A has full rank n.

But this is not enough
$\mathbb{E}(x \otimes x \otimes x)$ might be 0. e.g. if s_i are symmetric.

$$\mathbb{E}(x \otimes x \otimes x \otimes x) = \mathbb{E}\left(\otimes^4 x\right)$$

$$= \mathbb{E}\left(\otimes^4 As\right)$$

$$\mathbb{E}\left(\otimes^4 As\right)_{ijkl} = \mathbb{E}\left((As)_i (As)_j (As)_k (As)_l\right)$$

$$= \mathbb{E}\left(\sum_{i'} A_{ii'} s_{i'} \sum_{j'} A_{jj'} s_{j'} \sum_{k'} A_{kk'} s_{k'} \sum A_{ll'} s_{l'}\right)$$

$$= \sum_{i',j',k',l'} A_{ii'} A_{jj'} A_{kk'} A_{ll'}\, \mathbb{E}(s_{i'} s_{j'} s_{k'} s_{l'})$$

i', j', k', l'

$$\mathbb{E}(s_{i'} s_{j'} s_{k'} s_{l'}) = \begin{cases} \mathbb{E}(s_{i'}^4) & \text{if } i' = j' = k' = l' \\ \mathbb{E}(s_{i'}^2)\mathbb{E}(s_{k'}^2) & \text{if } i' = j' \,(\neq)\, k' = l' \\ & \text{or } i' = k',\ j' = l' \\ & \quad i' = l',\ j' = k' \\ 0 & \text{o.w.} \end{cases}$$

$$= \sum_{i'} A_{ii'} A_{ji'} A_{ki'} A_{li'} \mathbb{E}(s_{i'}^4)$$

$$+ \sum_{i' \neq k'} A_{ii'} A_{ji'} A_{kk'} A_{lk'}$$

$$+ \sum_{i' \neq j'} A_{ii'} A_{jj'} A_{ki'} A_{lj'}$$

$$+ \sum_{i' \neq l'} A_{ii'} A_{jl'} A_{kl'} A_{li'}$$

$$= \sum_{i'} A_{ii'} A_{ji'} A_{ki'} A_{li'} \left(\mathbb{E}(s_i^4) - 3\right)$$

$$+ \sum_{i', k'} A_{ii'} A_{ji'} A_{kk'} A_{lk'}$$

$$+ \sum_{i', j'} \qquad + \sum_{i', l'} .$$

$$\mathbb{E}(x_i x_j)\mathbb{E}(x_k x_l) + \mathbb{E}(x_i x_k)\mathbb{E}(x_j x_l) + \mathbb{E}(x_i x_l)\mathbb{E}(x_j x_k) = M_{ijkl}$$

So,

$$\boxed{\mathbb{E}(\otimes^4 x) - M = \sum_i \left(\mathbb{E}(s_i^4) - 3\right) A^{(i)} \otimes A^{(i)} \otimes A^{(i)} \otimes A^{(i)}}$$

A tensor decomposition into orthogonal vectors!
We can solve this by Tensor power iteration.

OR. pick random $v \sim N(0, I)$

Apply $$T[\cdot,\cdot,v,v] = \sum_i \underbrace{\left(\mathbb{E}(s_i^4) - 3\right)\left(A^{(i)} \cdot v\right)^2}_{\alpha_i} A^{(i)} A^{(i)T}$$

$$= A \begin{bmatrix} \alpha_1 & & 0 \\ & \ddots & \\ 0 & & \alpha_d \end{bmatrix} A^T$$

eigenvectors are columns of A!
eigenvalues are distinct WHP. !