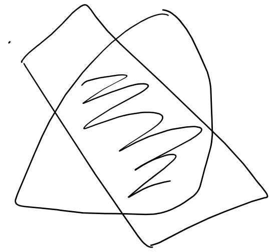
The Localization Method for High-dimensional Inequalities (TUTORIAL)

① Slicing (Bourgain '86) convex body $K \subseteq \mathbb{R}^n$ $x, y \in K$ $[x,y] \subseteq K$.

$Vol(K) = 1$ $\exists$ hyperplane H s.t. $Vol_{n-1}(K \cap H) \geq c$.

$\exists c > 0$



Thm [Klartag-Lehec; Bizeul]

② Thin-shell (Attila; Ball, Perissinaki; Bobkov-Koldobsky '03)

ν in $\mathbb{R}^n$ logconcave f is logconcave

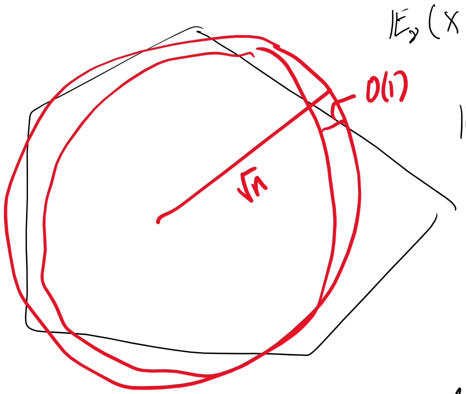
$f: \mathbb{R}^n \to \mathbb{R}_+$ $\forall x, y \in \mathbb{R}^n$, $\lambda \in [0,1]$

$f((1-\lambda)x + \lambda y) \geq f(x)^{1-\lambda} f(y)^{\lambda}$.

ν is isotropic if $\mathbb{E}_\nu(x) = 0$

$\mathbb{E}_\nu(x x^T) = I$

any dist with bonded second moments can be made isotropic by affine transformation



$\mathbb{E}(\|x\|^2) = n$ $\mathbb{E}_\nu((\|x\| - \sqrt{n})^2) = O(1)$.

③ ~~Large Deviatio~~ Large Deviation $t \geq 1$

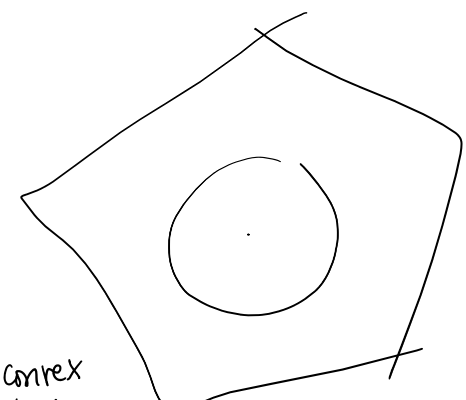
Thm (Paouris '06) $Pr_\nu(\|x\| > t\sqrt{n}) \leq e^{-ct\sqrt{n}}$.

isotropic, logconcave.

④ $\boxed{\text{conj.}}$ small-ball probability

$\varepsilon < \varepsilon_0$ $Pr_\nu(\|x\| \leq \varepsilon\sqrt{n}) \leq \varepsilon^{cn}$

Thm (Bizeul) ↑.



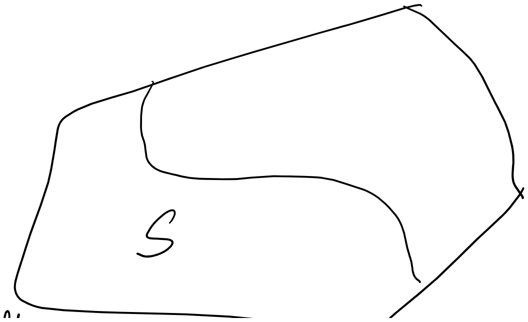
⑤ KLS conjecture. [KLS 95] isotropic, convex body K

$\exists c > 0 \; \forall S \subseteq K$ $Vol(S) \leq \frac{1}{2} Vol(K)$

$Vol_{n-1}(\partial S) \geq c \, Vol(S)$

ν : logconcave

$\nu(\partial S) \geq \frac{1}{\sqrt{\|A\|_{op}}} \nu(S)$



$\nu(\partial S) \approx \frac{\quad}{\sqrt{\|A\|_{op}}}$

Hyperplane cuts are within $O(1)$ of the optimal isoperimetric subset.

$$\Big(\psi_{KLS}^{(\nu)} = \sup \frac{\nu(S)}{\nu(\partial S)}$$

Thm [KLS95] $\nu(\partial S) \geq \frac{c}{\sqrt{\mathrm{tr}(A)}} \min\{\nu(S), \nu(\mathbb{R}^n \setminus S)\}$

$$\sqrt{\mathrm{tr}(A)} = \sqrt{\mathbb{E}(\|x\|^2)} \leq \mathrm{Diam}$$

ν has a Poincaré Inequality with "constant" C_{PI} if $\forall$ smooth f

$$\boxed{\mathrm{Var}_\nu(f) \leq C_{PI}\, \mathbb{E}_\nu(\|\nabla f\|^2)}$$

For logconcave ν: $\psi_{KLS}^2 \lesssim C_{PI} \underset{\nu}{\lesssim} \psi_{KLS}^2$

$\underline{\text{KLS}} \Rightarrow$ thin-shell $\Rightarrow$ slicing

$\Updownarrow$

small-ball

$\underline{\text{Thm}}$ (Klartag 23) $\psi_{KLS} = O(\sqrt{\log n})$

$C_{PI} \lesssim \log n$.

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(6) ν is t-strongly logconcave $\quad \nu(x) = e^{-t\frac{\|x\|^2}{2}} \cdot \nu_1(x)$

$\underline{\text{Thm}}$. $\frac{\nu(\partial S)}{\min \nu(S), \nu(\mathbb{R}^n \setminus S)} \gtrsim \sqrt{t} \iff C_{PI} \lesssim t$.

[Brascamp-Lieb] $\mathrm{Var}_\nu(f) \leq \mathbb{E}_\nu\left(\langle \nabla f, (-\nabla^2 \log f)^{-1} \nabla f\rangle\right)$

$\leq t\, \mathbb{E}(\|\nabla f\|^2)$

$\underline{\text{Thm}}$. $\nu(S) \leq \frac{1}{2}$

$$\frac{\nu(\partial S)}{\nu(S)\sqrt{\log \frac{1}{\nu(S)}}} \gtrsim \sqrt{t} \qquad \text{"log-Cheeger"}.$$

Log-Sobolev inequality (LSI) $\quad \nu$ satisfies LSI with "constant" C_{LSI}

$$\mathrm{Ent}_\nu(f) \leq C_{LSI}\, \mathbb{E}(\|\nabla f\|^2)$$

$C_{LSI} = O(1) \Rightarrow$ log-cheeger holds with constant $\Omega(1)$.

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(6) Certifiable hypercontractivity $\quad f(x) = \langle\langle 1, x\rangle^{\otimes k}, A\rangle$

$\mathrm{Var}_\nu(f) \lesssim (C_1 k)^{2k} \|A\|_F^2 \quad$ isotropic, logconcave ν

$\mathrm{Var}_\nu(f) \lesssim (C,K)\,\|A\|_F$ isotropic, logconcave

[KS] $C_1 \lesssim C_{PI}$

(7) Polynomial anticoncentration

[Carbery-Wright] deg d $p: \mathbb{R}^n \to \mathbb{R}$ on a convex body $K \subseteq \mathbb{R}^n$.

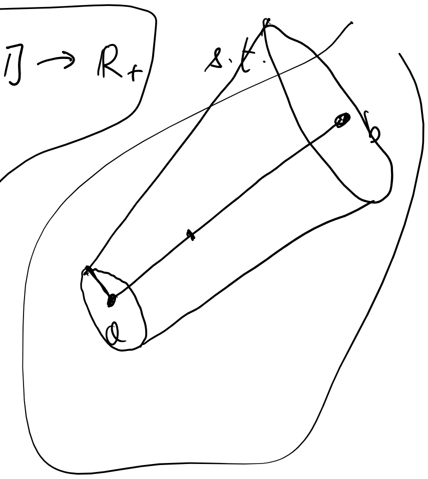
$d \le n$ $\quad \mathrm{Var}_K(p(x)) = 1 \Rightarrow \underset{\forall t}{\forall \varepsilon \geq 0} \quad \Pr_K(|p(x) - t| \le \varepsilon) \lesssim d\,\varepsilon^{\frac{1}{d}}$

Lemma [LS93] $f, g: \mathbb{R}^n \to \mathbb{R}$ lower semi-continuous

$$\int_{\mathbb{R}^n} f > 0 \qquad \int_{\mathbb{R}^n} g > 0$$

Then $\exists$ needle: $a, b \in \mathbb{R}^n$ linear functn $\ell: [0,1] \to \mathbb{R}_+$ s.t.

$$\int_N f = \int_0^1 f((1-t)a + tb)\, \ell(t)^{n-1}\, dt$$



s.t. $\int_N f > 0 \quad \int_N g > 0$.

Lemma [KLS95] $f_1, f_2, f_3, f_4: \mathbb{R}^n \to \mathbb{R}$ continuous.

The following are equivalent:

(1) $\forall$ logconcave $F: \mathbb{R}^n \to \mathbb{R}_+$ with compact support

$$\int F f_1 \int F f_2 \le \int F f_3 \int F f_4$$

(2) $\forall$ exp. needle $a, b \in \mathbb{R}^n$ $\gamma \in \mathbb{R}$ $\int_E f = \int_0^{|b-a|} f(a + t(b-a))\, e^{\gamma t}\, dt$

$$\int_E f_1 \int_E f_2 \le \int_E f_3 \int_E f_4$$

Lemma [Fradelizi-Guedon] K: convex body in $\mathbb{R}^n$ $\quad f$: upper semi-continuous

$P_f = \{ \mu \text{ logconcave} : \int_K f\, d\mu \ge 0 \}$. Then Extreme measures are

$P_f = \{ \mu \text{ logconcave} : \int_K f \, d\mu \geq 0 \}$. Then Extreme measures are

(1) Dirac x : $f(x) \geq 0$

(2) ν s.t. $\nu(x) \propto e^{\ell(x)} \mathbb{1}_{[a,b]}$ $\quad [a,b] \subseteq K$

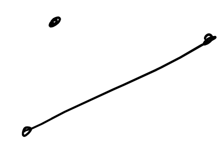
$\int f \, d\nu = 0 \quad \int_a^x f \, d\nu > 0$ for $x \in (a,b)$ or $\int_x^b f \, d\nu > 0$

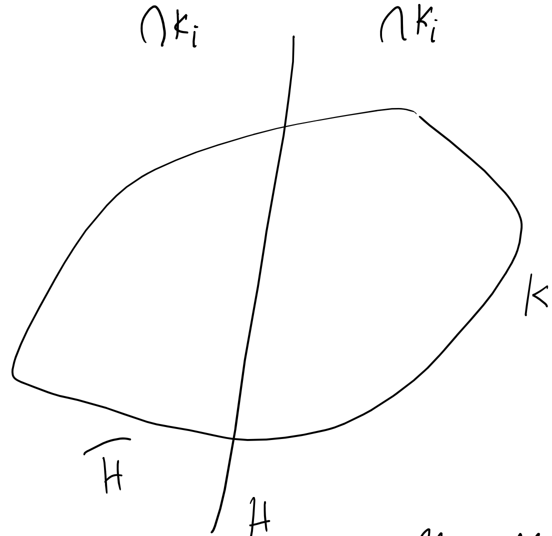
convex ϕ $\quad \operatorname{argmax}_{\mu \in P_f} \phi$

$\int f > 0 \quad \int g > 0 \qquad K_0$: huge ball.

$\exists$ sequence of convex bodies $K_0 \supseteq K_1 \supseteq K_2 \; \ldots$

$\int_{\cap K_i} f > 0 \quad \int_{\cap K_i} g > 0 \quad \lim_{i \to \infty} \bigcap K_i \rightarrow$

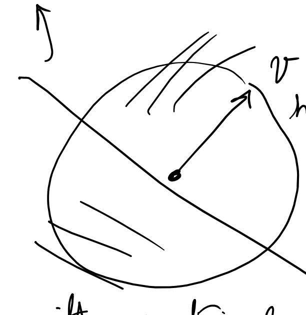




$\int_{K \cap H} g = \int_{K \cap \bar{H}} g > 0$

$v \perp A_{n-2}$

Claim $\exists H$

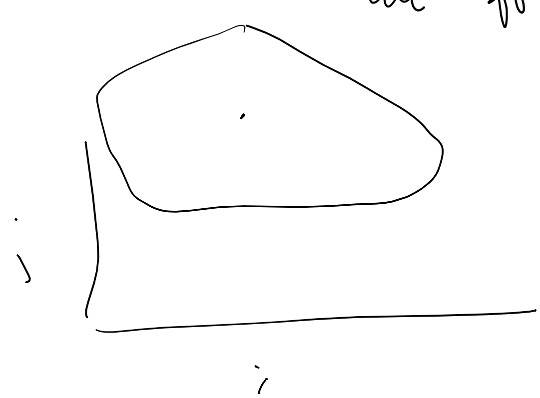


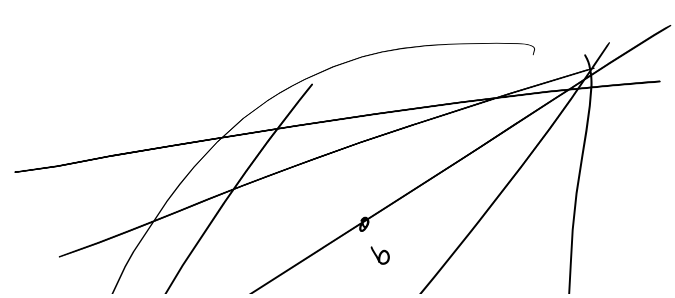
$h(v) = \int_{K \cap H_v} g - \int_{K \cap \bar{H}_v} g$

$h(-v) = -h(v)$

all affine subspaces with rational coordinates

$\{ x_i = r_1 \; , \; x_j = r_2 \}$





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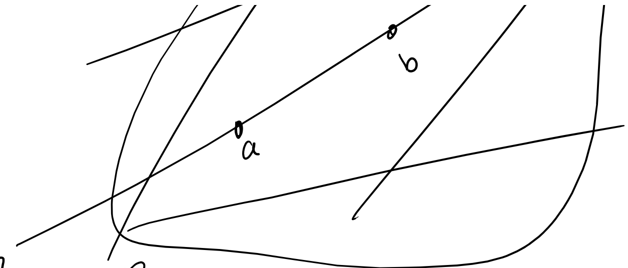
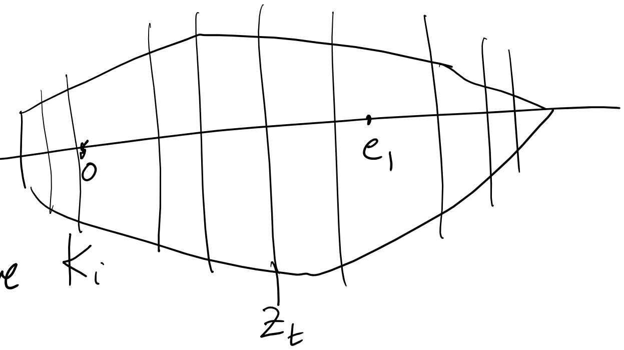


Claim $\exists$ concave $\psi : [0,1] \to \mathbb{R}_+$ st.

$$\int_0^1 f\, \psi(t)^{n-1} \geq 0 \qquad \int_0^1 g\, \psi(t)^{n-1} \geq 0$$

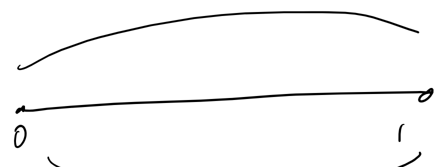
$[a,b] = [0, e_1]$

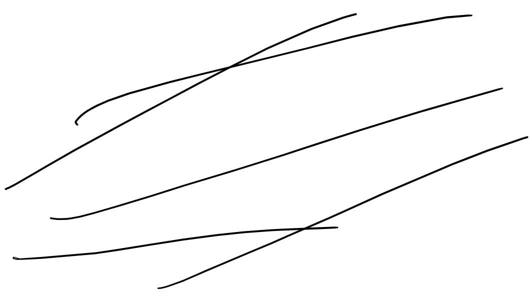
$$\psi_i(t) = \left(\frac{\mathrm{vol}_{n-1}(K \cap Z_t)}{\mathrm{vol}(K)}\right)^{\frac{1}{n-1}}$$

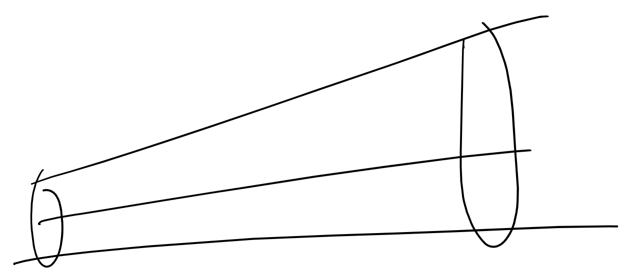


Brunn-Minkowski $\Rightarrow \psi_i$ is concave

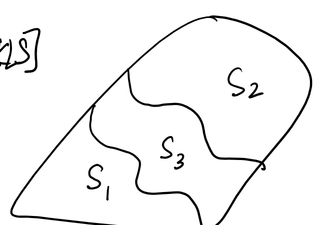
$\lim_{i\to\infty} \psi_i \to \psi$







Thm [KLS]



S_1, S_2, S_3 partition of $K \subseteq \mathbb{R}^n$

$\mathrm{Diam}(K) = D$

$$\mathrm{vol}(S_3) \geq \frac{2}{D}\, d(S_1, S_2) \min\{\mathrm{vol}(S_1), \mathrm{vol}(S_2)\}$$

Suppose not. Then

$\alpha\, \mathrm{vol}(S_1) > \mathrm{vol}(S_2)$

$\alpha\, \mathrm{vol}(S_2) > \mathrm{vol}(S_3)$

$\longrightarrow$ α

$f = \alpha \mathbb{1}_{S_1} - \mathbb{1}_{S_3}$

$g = \alpha \mathbb{1}_{S_2} - \mathbb{1}_{S_3}$

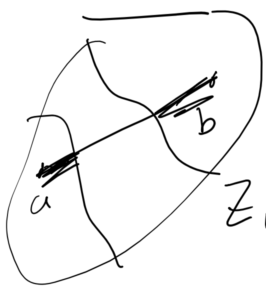
$\alpha \operatorname{vol}(S_2) > \operatorname{vol}(S_3)$ $\quad\Big|\longrightarrow\quad$ $g = \alpha \mathbb{1}_{S_2} - \mathbb{1}_{S_3}$

α

$\int f \,,\ \int g > 0 .$ $\qquad \dfrac{2\, d(\bar S_1, \bar S_2)}{|a-b|} \geq \dfrac{2}{D}\, d(S_1, S_2)$

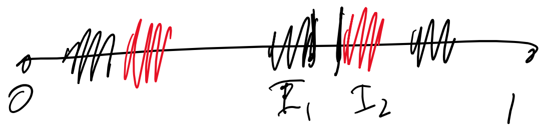
Apply LL. $\exists$ needle $a, b \in \mathbb{R}^n$ $\ell(t)$ s.t.

$$\int_0^1 f((1-t)a + tb)\, \ell(t)^{n-1}\, dt > 0 \qquad \int g\, \ell(t)^{n-1} > 0$$

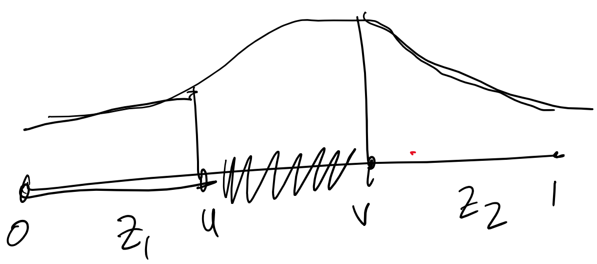


$$Z_i = \{ t \in [0,1] : (1-t)a + tb \in S_i \}$$

$$\int_{Z_3} \ell(t)^{n-1} dt < \alpha \int_{Z_1} \ell(t)^{n-1} ,\ \alpha \int_{Z_2} \ell(t)^{n-1}$$



Claim suffices to prove the contrary for $Z_1, Z_2\ Z_3$ single intervals.



$$\int_u^v f \geq 2|u-v| \min \int_0^u f ,\ \int_v^1 f$$

$\ell(t)^{n-1}$ is logconcave $\qquad$ unimodal f $\qquad \geq |u-v|$

$$\int_u^v f \geq |u-v| \min f(u), f(v) \qquad f(u) \leq f(v)$$

$$\int_0^u f \leq u f(u) \qquad \int_u^v f \geq \frac{|u-v|}{u} \cdot u f(u) \geq \frac{|u-v|}{u} \int_0^u f \geq |u-v|$$

Part II: Stochastic localization

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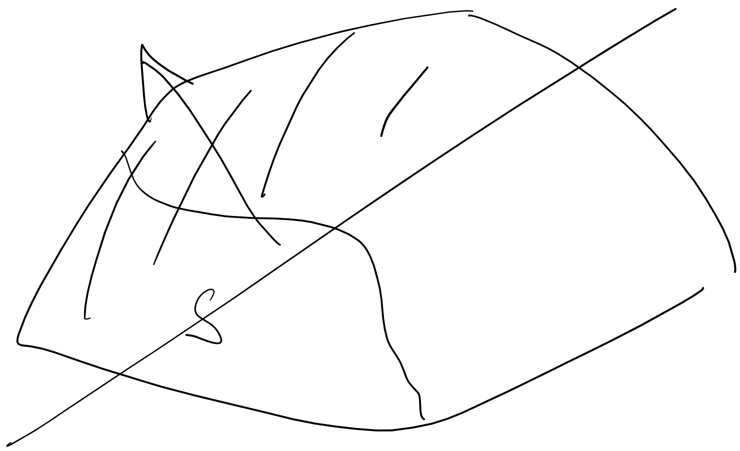


$$\boxed{\frac{vol(S)}{vol(K)} = c}$$

$$vol(\partial S) \geq c\, vol(S)$$

Start with an isotropic logconcave measure $p_0 = p$ $\quad \mathbb{R}^n$

$t \geq 0 \qquad p_t \qquad \mu_t = \mathbb{E}_{p_t}(x) = \int x\, p_t(x)\, dx$

$$(*) \quad \boxed{dp_t(x) = \langle x-\mu_t, dW_t \rangle\, p_t(x)}$$

dW_t Brownian Motion

$W_t = N(0, tI)$

$\mathbb{E}\left(dp_t(x)\right) = 0 \qquad \mathbb{E}\left(p_t(x)\right) = p_0(x)$ Martingale

$\mathbb{E}(W_t^2) = t \qquad dW_t = W_{dt}$

$\mathbb{E}(dW_t)^2 = \mathbb{E}(W_{dt})^2 = dt$

Itô:

$$dX_t = b_t\, dt + \sigma(t)\, dW_t$$

$$df(X_t) = \langle \nabla f(X_t), dX_t \rangle + \frac{1}{2} \langle \nabla^2 f(X_t), \sigma(t)\sigma(t)^T \rangle\, dt$$

$$(*) \quad dp_t(x) = p_t(x)(x-\mu_t)^T dW_t$$

$$d \log p_t(x) = \frac{1}{p_t(x)} dp_t(x) - \frac{1}{2} \frac{\|x-\mu_t\|^2 \cancel{p_t(x)^2}}{\cancel{p_t(x)^2}} dt$$

$$= (x-\mu_t)^T dW_t - \frac{1}{2}\|x-\mu_t\|^2 dt$$

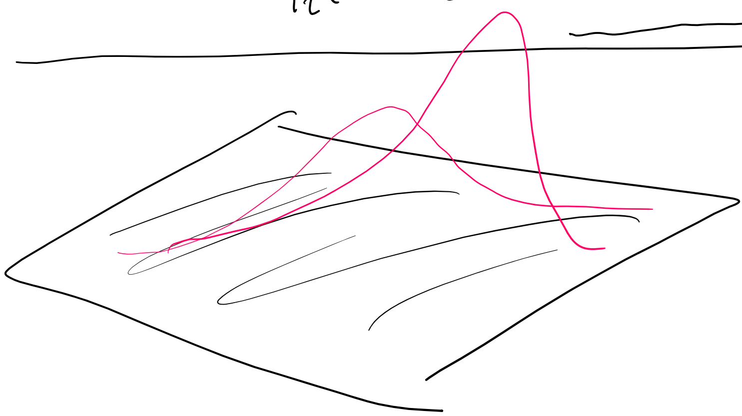
$$= -\frac{1}{2}\|x\|^2 dt + x^T(\mu_t\, dt + dW_t) - \underbrace{\mu_t^T dW_t - \frac{1}{2}\|\mu_t\|^2 dt}$$

$$= -\frac{1}{2}\|x\|^2 dt + x^T(\mu_t\, dt + dW_t) - \underbrace{\mu_t^T dW_t - \frac{1}{2}\|\mu_t\|^2 dt}_{g(t)}$$

$$= -\frac{1}{2}\|x\|^2 dt + x^T(\mu_t\, dt + dW_t) - g(t)\, dt$$

$$\log \frac{p_t(x)}{p_0(x)} = -\frac{t}{2}\|x\|^2 + c_t^T x - \bar{g}(t)$$

$$p_t(x) \propto e^{-\frac{t}{2}\|x\|^2 + c_t^T x} \cdot p_0(x)$$



p_t is t-strongly logconcave.

$C_{PI} \le t$

$\mathrm{cov}(p_t) \preceq \frac{1}{t} I$

Example   Isoperimetry   Goal: bound C_{PI}, ψ_{KLS} of $p_0 = p$

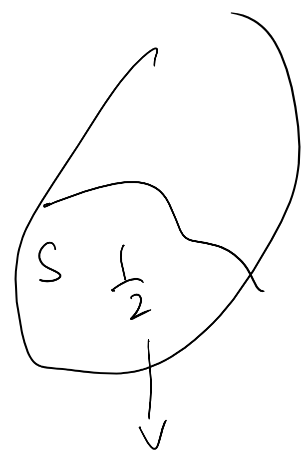
Fix subset S. $p_0(S) = \frac{1}{2}$ Goal: bound $\frac{p_0(\partial S)}{p_0(S)}$

Apply (SL) $p_0 \longrightarrow p_t$ $\frac{p_t(\partial S)}{p_t(S)} \gtrsim \sqrt{t}$

$p_0(\partial S) = \mathbb{E}(p_t(\partial S))$ | $p_0(S) = \mathbb{E}(p_t(S))$

(by C_{PI}) $\left(\gtrsim \sqrt{t}\, \mathbb{E}(p_t(S))\right)$

$\gtrsim \sqrt{t}\, \Pr\left(\frac{1}{4} \le p_t(S) \le \frac{3}{4}\right)$



$\frac{1}{4} \le p_t(S) \le 3/4$

$$d(p_t(S)) = d\left(\int_S p_t(x)\, dx\right) = \int_S dp_t(x)\, dx$$

$$= \int_S \langle x - \mu_t, dW_t\rangle\, p_t(x)\, dx$$

$- \int_S \langle x - \mu_t, dW_t \rangle \, p_t(x) \ldots$

$$d p_t(S) = \underbrace{\left(\int_S (x-\mu_t)\, p_t(x)\, dx \right)^T}_{\sigma(t)} dW_t$$

$$\boxed{\sigma(t)\sigma(t)^T dt}$$

$$d[p_t(S)]_t = \left\| \int_S (x-\mu_t)\, p_t(x)\, dx \right\|^2 dt$$

$$\leq \sup_{\|v\|=1} \left(\int_S v^T(x-\mu_t)\, p_t(x)\, dx \right)^2 dt \leq \sup_v \int_S \left(v^T(x-\mu_t)\right)^2 p_t(x)\, dx\, dt$$

$$\int_S p_t(x)\, dx$$

$$\leq \sup_{\|v\|=1} v^T \left(\int_{R^n} (x-\mu_t)(x-\mu_t)^T p_t(x)\, dx \right) v \, dt$$

$$\leq \|A_t\|_{op}\, dt$$

$$[p_t(S)]_t \leq \sup_{s\leq t} \|A_s\|_{op} \cdot t$$

M_t Martingale

$\forall u>0$, σ^2

$$\Pr\left(\exists t>0,\ M_t > u,\ [M_t]_t \leq \sigma^2\right) \leq e^{-\frac{u^2}{2\sigma^2}}$$

Lem. For $t \leq t_0 = \frac{c}{(\log n)^2}$ $\Pr\left(\exists s \leq t_0 : \|A_s\|_{op} > 2\right) \leq e^{-\frac{1}{c t_0}}$

β

$$\frac{1}{\beta} \ln \operatorname{tr} e^{\beta A_t}$$

$$|p_t(S) - p_0(S)| \leq 4 t_0$$

$$\frac{p_0(\partial S)}{p_0(S)} \gtrsim \sqrt{t_0} = \frac{1}{\log n}$$

$\psi_{KLS} = O(\log n)$ $\qquad$ $PI: O(\log^2 n)$

$\sqrt{\log n}$ $\qquad$ $\log n$

$$t_0 = \frac{c}{\log n}$$

(1) Classical via SL $\qquad$ (2) Slicing $\Longleftrightarrow$ Small-ball prob $\quad$ cn

① Classical via SL ② Slicing $\Longleftrightarrow$ small-ball prop

(SB) $\Pr(\|X\| \le \varepsilon\sqrt{n}) \le \varepsilon^{cn}$

Prop [Bizeul '25] For a $\Omega(1)$-strongly logconcave p_t
$\operatorname{tr}(A_t) = \Omega(n)$ $\Longrightarrow$ (SB) holds.

$t = c \to \Omega(1)$ strongly logconcave p_t.

$$\boxed{\text{Thm (Guan '24) WHP } \operatorname{tr}(A_t^2) = O(n) \text{ for } t \to c.}$$

Lemma For any $\lambda > 1$, w.p. $1 - \frac{1}{\lambda}$ $\forall S$ $p_0(S) \le e^{cnt} (p_t(S))^{\frac{1}{\lambda}}$

$S = \varepsilon\sqrt{n} B_n$ ε^{cn}

Pf. $g_t = p_t(S)$ $dg_t = \int_S \langle x - \mu_t, dW_t \rangle p_t(x)\,dx$

$$d[g_t]_t = \left\| \int_S (x-\mu_t) p_t(x)\,dx \right\|^2 dt$$

$$\le D^2 g_t^2\,dt \quad D = O(\sqrt{n})$$

$$\le c n g_t^2\,dt.$$

$$\mathbb{E}\, d\log g_t \ge \mathbb{E} \underbrace{\frac{1}{g_t} dg_t}_{0} - \frac{1}{\cancel{g_t^2}} c n \cancel{g_t^2}\,dt$$

$$\mathbb{E} \log g_t \ge \log g_0 - cnt$$

$$\log g_0 \le \frac{1}{\lambda} \log g_t + cnt$$

$$\boxed{g_0 \le e^{cnt} (g_t)^{\frac{1}{\lambda}}}$$

Markov to $\log \frac{1}{g_t}$

w.p. $1 - \frac{1}{\lambda}$

$$\log \frac{1}{g_t} \le \lambda\, \mathbb{E}\left(\log \frac{1}{g_t}\right)$$

$$\frac{1}{\lambda} \log g_t \ge \mathbb{E}(\log g_t)$$

$$\boxed{g_0 \le e^{cnt} (g_t)^{\frac{1}{\lambda}}}$$

$\le e^{cn}$

$$dp_t(x) = \langle x - \mu_t, dW_t \rangle \, p_t(x) \qquad hI$$

$$p_{t+1}(x) = p_t(x) \left(1 + \sqrt{h} \langle x - \mu_t, Z \rangle \right) \qquad Z \sim N(0, I)$$