CS 4540/CS 8803: Algorithmic Theory of Intelligence Fall 2023

Lecture 9: Fixed Points and Games

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1 Fixed Point Theorems

Let S be a state space, and $T : S \mapsto S$ be a transition function. Consider the following sequence: $x_0, T(x_0), T(T(x_0)), \ldots$. Does this converge to a fixed point with $T(x) = x$?

In general, it's hard to say whether T has a fixed point. However, we can characterize some sufficient conditions.

1.1 Useful Definitions

Definition 1 (Contractive Function)**.** *Let S be a set with a metric d (think, for example, Euclidean distance). A function $T : S \mapsto S$ is contractive with respect to d if there exists an $r \in (0, 1)$ such that, for all $x, y \in S$, $d(T(x), T(y)) \leq rd(x, y)$.*

Definition 2 (Convex Set)**.** *Let $S \subset \mathbb{R}^n$; S is convex if for any $x, y \in S$ and $t \in [0, 1]$, then $tx + (1 - t)y \in S$.*

Definition 3 (Compact Set)**.** *Let $S \subset \mathbb{R}^n$. S is compact if it is both closed and bounded (there exists a finite ball B where $S \subset B$).*

1.2 Sufficient conditions for convergence

Theorem 4 (Banach's fixed point theorem)**.** *Let S be a closed set with a metric d. Let T be continuous and contractive (Definition 1) in d. Then, T converges to a unique fixed point.*

We will prove the theorem for the case where $S = [0, 1]$, but it holds in general.

Proof. First: Existence
Suppose for contradiction that there is no fixed point. Then, $T(0) > 0$ and $T(1) < 1$.

Define $f(x) = T(x) - x$. So $f(0) > 0$, and $f(1) = T(1) - 1 < 0$. f is still continuous, so by the intermediate value theorem, $\exists x^* : f(x^*) = 0$. For this x^*, $T(x^*) = x^*$.
Second: Convergence
Consider a sequence, starting with x_0, where $x_1 = T(x_0)$, and for any $k > 0$, $x_k = T(x_{k-1})$

For any $k > 0$, by the contraction property,

$$d(x_{k+1}, x_k) = d(T(x_k), T(x_{k-1})) \leq rd(x_k, x_{k-1}) \leq r^k d(x_0, x_1)$$

Since $r < 1$, this converges to 0, so the sequence must approach a fixed point.
Third: Uniqueness
Suppose for contradiction that there are two fixed points, x^* and y^*. By the contractive property, $d(T(x^*), T(y^*)) < rd(x^*, y^*)$. However, since they are fixed points, $d(x^*, y^*) = d(T(x^*), T(y^*))$. The only way that this can hold is if $d(x^*, y^*) = 0$; since d is a metric, this is only true if $x^* = y^*$. □

Theorem 5 (Brouwer's Fixed Point Theorem)**.** *Let $f : S \mapsto S$ be a continuous function, where S is a compact, convex set in $\mathbb{R}^n$. Then, f has a fixed point.*

We will prove the theorem when S is the n-dimensional simplex, where $S = \Delta_n = \{x \in \mathbb{R}^{n+1} : x_i \geq 0, \sum_i x_i = 1\}$, but it holds in general.

Proof. The main tool for the proof will be the following combinatorial lemma.

Lemma 6 (Sperner's Lemma)**.** *Let S be a n-dimensional simplex, and consider any triangulation of S. Color the vertices in $n+1$ colors using the following constraints: each corner of S has a different color, if a vertex v is on a sub-face of S, then its color matches one of the vertices of its face.*

Then, there are an odd number of small simplices with exactly $n+1$ colors.

To illustrate the lemma in in $\mathbb{R}^2$: a 2-dimensional simplex is a triangle. A triangulation divides S into some number of smaller triangles. At each vertex of a smaller triangle, we assign it a color in {red, black, white}. The only constraint is that the three vertices of the big triangle have different colors, and any vertex on the edge can only choose between two colors. An example is shown in Figure 1.

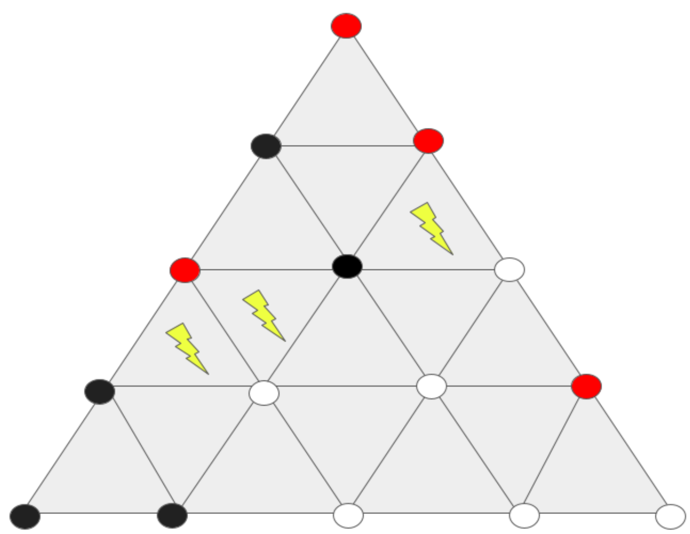

Figure 1: An example of a Sperner coloring on a triangle. The lightning bolts show the three rainbow triangles.

Proof of Sperner's Lemma. To prove this lemma for 2-D: Consider a graph $G = (V, E)$. V has one vertex for each of the small triangles, plus one vertex representing the 'outside'. Two vertices has an edge in the graph if the two corresponding triangles share one red-black edge.

The vertex representing 'outside' has odd degree - try to think about why!

A triangle cannot have three red-black edges. So, any triangle with odd degree must be a 'rainbow' triangle, with one red-black edge, one red-white edge, and one black-white edge.

Any simple graph must have an even number of odd-degree vertices. So, there must be an odd number of rainbow triangles. $\square$

Now, we can use this idea to prove the main theorem.

Let $x \in S$ be any point on the simplex. Since $f(x) \in S$, $\sum_i f(x)_i = \sum_i x_i = 1$. Therefore, there exists a $j : f(x)_j \leq x_j$, $x_j > 0$.

Now, pick a triangulation of S, and color it in the following way: let x be one of the vertices. Let j be the first index such that $f(x)_j \leq x_j$. Color x with the jth color.

No matter the triangulation, there exists a simplex that has all $n+1$ colors, so $\forall j \in [n+1]$, there is a vertex v^j of the simplex labeled by the j'th color, i.e., $f(v^j)_j \leq v^j_j$. Since the triangulation can be made arbitrarily fine, these $n+1$ vertices can be arbitrarily close to each other. Taking the limit, the triangle converges to a point, which must be a fixed point of f since the only way that $f(x)_j \leq x_j \forall j \in [n+1]$ is when $x = f(x)$. $\square$

2 Game Theory

2.1 Two classic examples

Game theory is the study of mathematical models of strategic interactions among rational agents. We firstly give two examples here.

Prisoner's dilemma. Say there are two prisoners A and B. Each prisoner has two choices: keep silent or testify against the other. They cannot exchange messages with each other when making the choice. This leads to four possible outcomes with the following consequences:

- If A and B both remain silent, they will each serve one year in prison.
- If A testifies against B and B remains silent, A will be free and B will serve three years in prison.
- If B testifies against A and A remains silent, B will be free and A will serve three years in prison.
- If A and B testifies against each other, they will each serve two years in prison.

Here the optimal strategy is to testify because whether or not the other prisoner testifies, you can serve fewer years in prison by testifying.

		B	
		Silent	Testify
A	Silent	1,1	3,0
	Testify	0,3	2,2

Years in Prison

Figure 2: Prison's dilemma.

		Odd player	
		Head	Tail
Even player	Head	1,-1	-1,1
	Tail	-1,1	1,-1

Win(1), Lose(-1)

Figure 3: Matching pennies.

Matching Pennies This game has two players: the even player and the odd player. Each player has a penny and will secretly turn the penny to head or tail. Then two players reveal their choices simultaneously. If the number of the heads of two pennies is odd, then the odd player wins. Otherwise the even player wins.

The optimal strategy here is to pick head or tail randomly with even probability. Otherwise, the opponent will choose the action with higher probability of winning.

2.2 Nash Equilibrium

Now we consider a general game, where we have 2 players A,B, and each of them has n possible actions (strategies). For each player, we define a *payoff matrix* as an $n \times n$ matrix, where the (i, j) entry is what the player gains if A takes action i and B takes action j. We denote the payoff matrix as A and B respectively. Moreover, we let x and y be mixed strategies of A and B respectively, i.e., they are probability distributions over the strategies: $x, y \in \Delta_n$ ($n-$dimensional simplex). If (x, y) form an equilibrium, this means that given B's strategy y, then by playing x, the payoff for A is as high as possible, and simultaneously given A's strategy x, by playing y, the payoff for B is as high as possible. In other words, x is a best response to y and y is a best response to x. We can represent this as

$$\forall i \in [n], x^\top Ay \geq (Ay)_i, \qquad \forall i \in [n], x^\top By \geq (B^\top x)_i.$$

To study the equilibrium of the game, we first define a game to be *explicit*, if the payoff matrix is explicit. Then we have the following theorem.

Theorem 7. *Every k-player explicit game has at least one Nash equilibrium (NE).*

It is tempting to apply Brouwer's fixed point theorem to the best response function which maps a pair (x, y) to $(br(x), br(y)$. If we could do this, then a fixed point would be a Nash equilibrium. However, best response is not a function, there potentially many possible best responses! So we need to define it more carefully.

Proof. We will only prove this for two-player games. For $i \in [n]$, we define

$$C_i^x(x,y) = \max\{0, (Ay)_i - x^\top Ay\}, \quad C_i^y(x,y) = \max\{0, (B^\top x)_i - x^\top By\}$$

We define a continuous function $f : \Delta_n \times \Delta_n \to \Delta_n \times \Delta_n$ as $f(x,y) = (f_1(x,y), f_2(x,y))$, where

$$f_1(x,y)_i := \hat{x}_i = \frac{x_i + C_i^x(x,y)}{\sum_{j=1}^n \left(x_j + C_j^x(x,y)\right)}, \quad f_2(x,y)_i := \hat{y}_i = \frac{y_i + C_i^y(x,y)}{\sum_{j=1}^n \left(y_j + C_j^y(x,y)\right)}$$

Clearly the image $f(x,y) \in \Delta_n \times \Delta_n$. Then by Brouwer's Fix Point Theorem, there exists $x, y \in \Delta_n$ such that $f(x,y) = (x,y)$. We will show that this is the optimal strategy. Suppose that x is not a best response. This implies that there exists $i \in [n]$ s.t. $(Ay)_i > x^\top Ay$. So we have $\sum_{k=1}^n C_k^x(x,y) > 0$.

If there is no j s.t. $C_j^*(x,y) = 0$. Then for all $j \in [n], C_j^*(x,y) > 0$, which implies that $(Ay)_j > x^\top Ay, 1 \leq j \leq n$. By summing up over all j, we have

$$x^\top Ay = (\sum_{j=1}^n x_j) x^\top Ay = \sum_{j=1}^n x_j x^\top Ay < \sum_{j=1}^n x_j (Ay)_j = x^\top Ay$$

So there exists $j \in [n]$ such that $C_j^*(x,y) = 0$. For this j, we have

$$f_1(x,y)_j = \frac{x_j + 0}{1 + \sum_{k=1}^n C_k^x(x,y)} < x_j$$

This is a contradiction and so (x, y) must be a Nash Equilibrium. Thus x is optimal. Similarly y is optimal as well. So (x, y) is the optimal mutual response.

$\square$