OPT from MEM (GRAD from EVAL)

Wednesday, January 29, 2020 6:16 PM

If f is differentiable with a continuous derivative, then we can write

$$\frac{\partial f(x)}{\partial x_i} = \lim_{h \to 0} \frac{f(x+he_i) - f(x)}{h}$$

$$= \frac{f(x+he_i) - f(x)}{h} + o(h)$$

So it takes $n+1$ calls:

$f(x), f(x+he_1) \cdots, f(x+he_n)$.

But what if f is not differentiable?

e.g. $f = \delta_K$.

Idea: Assume f is convex and L-Lipshitz.

i.e. $\forall x, y \quad |f(x) - f(y)| \le L|x-y|$.

Such a function is differentiable almost everywhere.

Lema. L-Lipshitz, convex $f : B(0,1) \to \mathbb{R}$

$$\mathbb{E}\left(\|\nabla^2 f\|_F\right) \leq nL.$$

Pf. $\nabla^2 f \succcurlyeq 0$ (where defined)

$$\Rightarrow \|\nabla^2 f\|_F = \sqrt{\Sigma \lambda_i^2} \leq \Sigma \lambda_i = \text{trace}(\nabla^2 f)$$

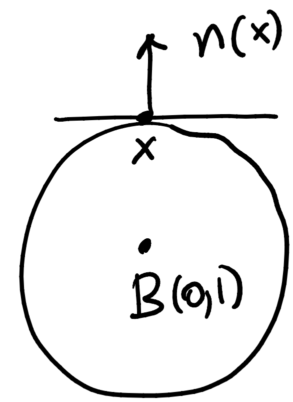
$$\therefore \int_{B(0,1)} \|\nabla^2 f(x)\|_F \, dx \leq \int_{B(0,1)} \text{tr}(\nabla^2 f(x))\, dx = \int_{B(0,1)} \sum_i \frac{\partial^2}{\partial x_i^2} f(x)\, dx$$

$\Delta f(x)$

"Laplacian".

Stokes thm:

$$\int_\Omega f' = \int_{\partial\Omega} f$$



$$\int_{B(0,1)} \Delta f(x)\, dx = \int_{\partial B(0,1)} \langle \nabla f(x), n(x) \rangle \, dx$$

$$\leq |\partial B(0,1)| \cdot L$$

$$\Rightarrow \mathbb{E}\, \|\nabla^2 f\|_F \leq \underline{|\partial B(0,1)|} \cdot L = nL.$$

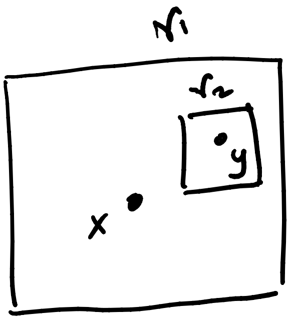
$$\Rightarrow \mathbb{E}_{B(0,1)} \|\nabla^2 f\|_F \le \frac{|\partial B(0,1)|}{|B(0,1)|} \cdot L = nL .$$

<u>Lemma.</u> $B_\infty(x,r) = \{ y : \|x-y\|_\infty \le r \}$

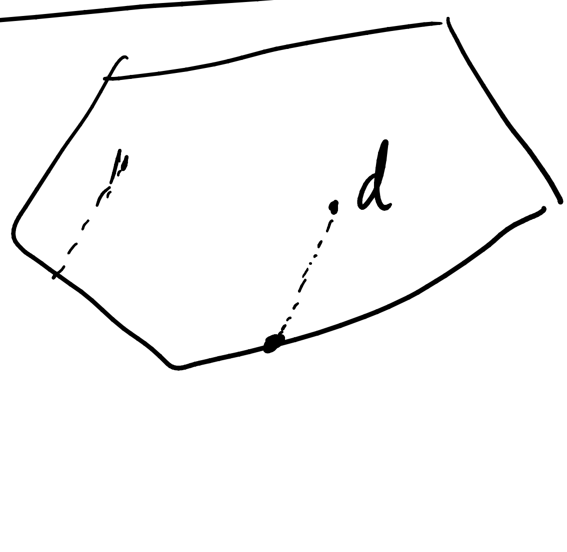
$\forall\ r_1 \ge r_2 \ge 0$, convex, L-Lipshitz F, $\forall z \in B_\infty(x, r_1+r_2)$

$$\mathbb{E}_{y \in B_\infty(x,r_1)} \mathbb{E}_{z \in B_\infty(y,r_2)} \|\nabla f(z) - g(y)\|_1 \le n^{3/2} \cdot \frac{r_2}{r_1} \cdot L$$

where $g(y) = \mathbb{E}_{B_\infty(y,r_2)} (\nabla f(y))$.



avg over r_2 box around y is close to <u>constant</u>.



Can be used to compute SEP.

$$\alpha_x(d) = \max \alpha$$
$$d + \alpha x \in K.$$

$\dot{x}$ $$h_x(d) = -\alpha_x(d)\,\|x\|_2$$

convex! Lipshits!

$0 \in K$: $\nabla_x h_x(0)$ is a separating normal!

How to compute: For each i

- Pick y at random in $B_\infty(x, r_1)$
- — z — $B_\infty(y, r_2)$

α z β e_i

$$g_i = \frac{f(\beta) - f(\alpha)}{2r}.$$

$O(n)$ EVALS.

Pf. Let $w_i(z) = \langle \nabla f(z) - g(y), e_i \rangle$ $\forall i \in [n]$.

$$\int_{B_\infty(y,r_2)} \|\nabla f(z) - g(y)\|_1 \, dz \leq \sum_i \int_{B_\infty(y,r_2)} |w_i(z)| \, dz$$

Poincare inequality: Ω connected, bounded.

$f : \Omega \to R$ smooth.

$$\left\| f - \frac{1}{|\Omega|}\int f \right\|_{L^1(\Omega)} \leq \sup \frac{2|S||\Omega \setminus S|}{|\partial(S)||\Omega|} \cdot \|\nabla f\|_{L^1(\Omega)}$$

$$\left\| f - \frac{1}{|\Omega|} \int_\Omega f \right\|_{L^1(\Omega)} \le \sup \frac{\cdots}{|\partial(S)||\Omega|} \| \cdots \|_{L(\Omega)}$$

For $\Omega = B_\infty(\cdot, r_2)$

$$\int_{B_\infty(y, r_2)} |w_i(z)| \, dz \le r_2 \int_{B_\infty(y, r_2)} \| \nabla w_i(z) \| \, dz$$

$$= r_2 \int_{B_\infty(y, r_2)} \| \nabla^2 f(z) e_i \|_2 \, dz$$

$$\therefore \sum_i \int_{B_\infty(y, r_2)} |w_i(z)| \, dz \le r_2 \cdot \sqrt{n} \int_{B_\infty(y, r_2)} \| \nabla^2 f \|_F \, dz$$

using Lemma 1,

$$\mathbb{E}_{z \in B_\infty(y, r_2)} \| \nabla f(z) - g(y) \|_1 \le r_2 \sqrt{n} \, \mathbb{E}_{B_\infty(y, r_2)} \Delta f(z) \, dz$$

$$= r_2 \sqrt{n} \, \Delta h(y)$$

$$h(y) = \frac{1}{(2r_2)^n} \cdot f * \chi_{B_\infty(0, r_2)}(y).$$ just the def

---

$$\int \cdots \int \langle \nabla f(\cdot), n(\cdot) \rangle \, du$$

$$\int_{B_\infty(x, r_1)} \Delta h(y)\, dy = \int_{\partial B_\infty(x, r_1)} \langle \nabla h(y), n(y) \rangle \, dy$$

f is L-Lipshitz $\Rightarrow$ h is L-Lipshitz

$\Rightarrow$

$$\left|\mathbb{E}_{B_\infty(x, r_1)} \Delta h(y)\right| \le \frac{1}{(2r_1)^n} \int_{\partial B_\infty(x, r_1)} \|\nabla h(y)\|_\infty \|n(y)\|_1$$

$$\le \frac{(2r_1)^{n-1}}{(2r_1)^n} \cdot 2n \cdot L \le \frac{nL}{r_1}$$

$\therefore$ Overall Bound $\le n^{3/2} \cdot \frac{r_2}{r_1} \cdot L$.

OPT $\longrightarrow$ MEM.

Thm. $F(x) = e^{-\alpha c^T x} \cdot 1_K(x)$

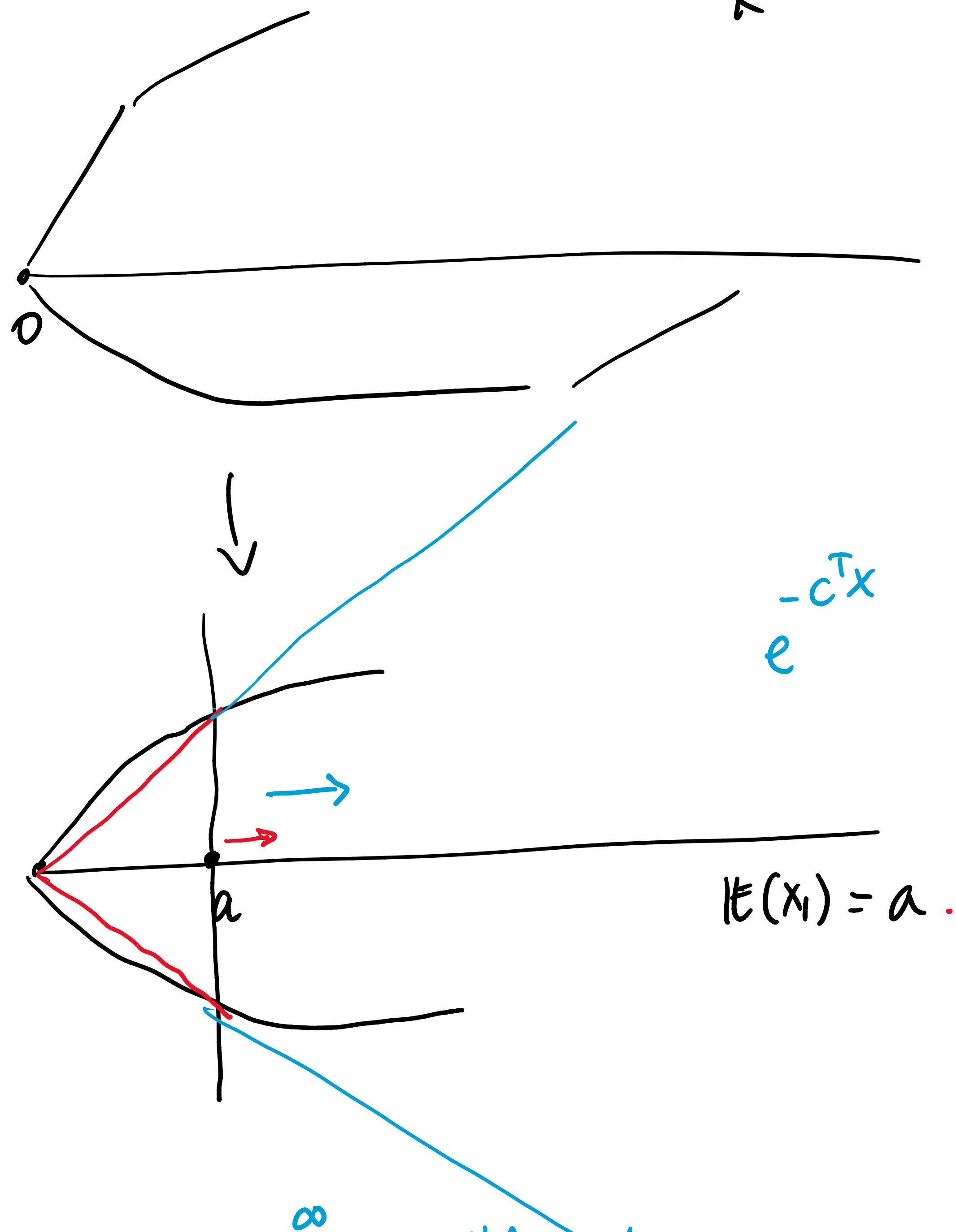
$\min c^T x$ is bounded.

$\ldots (c^T \ldots) \le \min c^T x + n$

... $c^T x$ is bounded.

$$\mathbb{E}(c^T x) \le \min_K c^T x + \frac{n}{\alpha}.$$

Pf. $c = e_1$ assume $\arg\min_K c^T x = 0$.



$$\mathbb{E}(c^T x) = \int^{\infty} y\, e^{-\alpha y}\, y^{n-1}\, dy$$

$$\mathbb{E}(c^T x) = \frac{\int_0 y\, e^{-\alpha y}\, y^{n-1}\, dy}{\int_0^\infty e^{-\alpha y}\, y^{n-1}\, dy}$$

$z = \alpha y$

$$= \frac{1}{\alpha} \frac{\int_0^\infty e^{-z} z^n \, dz}{\int_0^\infty e^{-z} z^{n-1}\, dz} = \frac{n!}{\alpha (n-1)!} = \frac{n}{\alpha}.$$

Claim $\int_0^\infty \underset{dv}{e^{-z}} \underset{u}{z^n} = n!$

$$= \left[-e^{-z} z^n\right]_0^\infty + n\int_0^\infty e^{-z} z^{n-1}$$

So sampling from $e^{-\frac{n}{\varepsilon} c^T x}$, $x \in K$

gives $\mathbb{E}(c^T x) \le OPT + \varepsilon$. !!

This applies to F approx. convex.

$$\max \left| f(x) - F(x) \right| \le \frac{\varepsilon}{n}$$

$$\max_{x \in K} |F'(x) - F(x)| = \frac{\varepsilon}{n}$$

we can find x: $F(x) \le \min_{x \in K} F(x) + O(\varepsilon)$.