Center of Gravity.

Monday, January 20, 2020 5:41 PM

Yudin

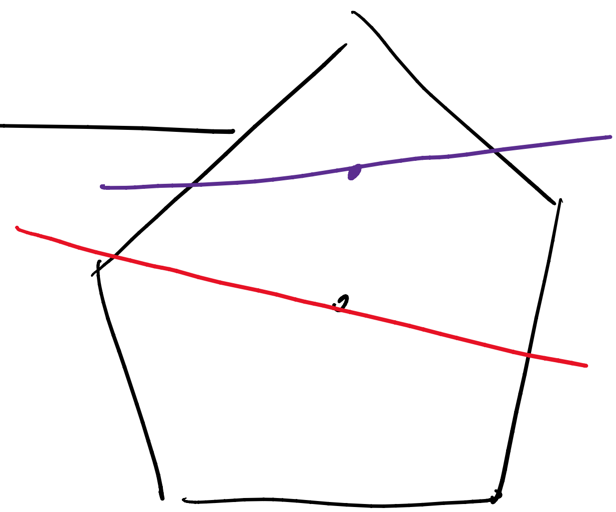
$E_0 = \text{Cube}[-R/2, R/2]^n$

$x^0 = 0$

Separating plane $a^T(x - x^k) \le 0$

$$\left[\begin{array}{l} E_{k+1} = E_k \cap H_k \\ x^{k+1} = CoG(E_{k+1}) = \frac{1}{\mathrm{Vol}(E_{k+1})}\int_{x\in E_{k+1}} x \, dx \end{array}\right.$$

Repeat.

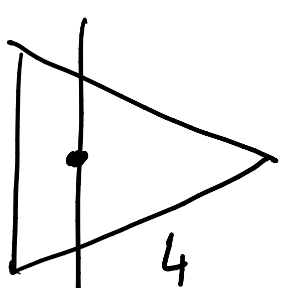


$\nu(E) = \mathrm{Vol}(E)^{\frac{1}{n}}$.

Clearly decreases.

At what rate?

in 2-d.



$\frac{5}{9}$ $\frac{4}{9}$

worrisome!

$\bar{q}$

Thm (Grunbaum) $\forall$ convex body K, any halfspace H containing center of gravity z of K satisfies

$$\mathrm{Vol}(K \cap H) \geq \frac{1}{e} \mathrm{Vol}(K).$$

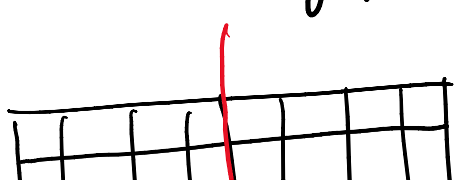
$\Rightarrow$ volume decreases by $\left(1 - \frac{1}{e}\right)$ or faster.

Thm. $O\left(n \log \frac{R}{r}\right)$ iterations suffice to reach set of volume r^n.

Thm. Given separation oracle for convex set K, and $K \subseteq R B_2^n$; and if K nonempty then $\exists x^0 : x^0 + r B_2^n \subseteq K$, Cutting Plane method finds $x \in K$ in $O\left(n \log \frac{R}{r}\right)$ iterations.

Thm. This is asymptotically the best possible.

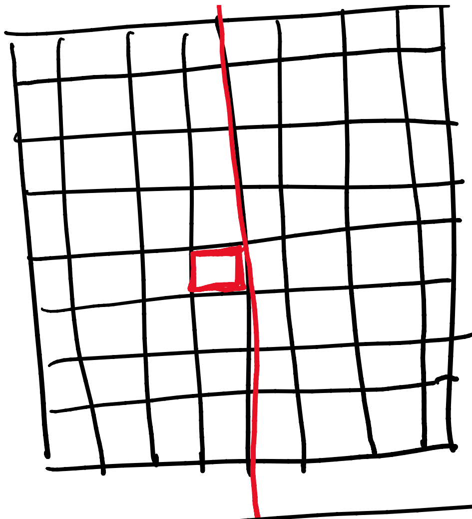
Pf.



Each iteration reduces

at most

II.



Each iteration reduces volume by at most $\frac{1}{2}$.

$\Rightarrow \Omega\left(n \log \frac{R}{r}\right)$ iterations.

Q1. How to prove volume decrease?

Q2. How to find CoG?

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f is logconcave if $\log f$ is concave, $f \geq 0$.

$$f(\lambda x + (1-\lambda) y) \geq f(x)^{\lambda} f(y)^{1-\lambda}.$$

e.g. $\mathbb{1}_K$, $e^{-\frac{\|x\|^2}{2}}$, $e^{-\frac{\|x\|^2}{2}} \cdot \mathbb{1}_K$

Lemma. product, min of logconcave f, g is logconcave. So is convolution.

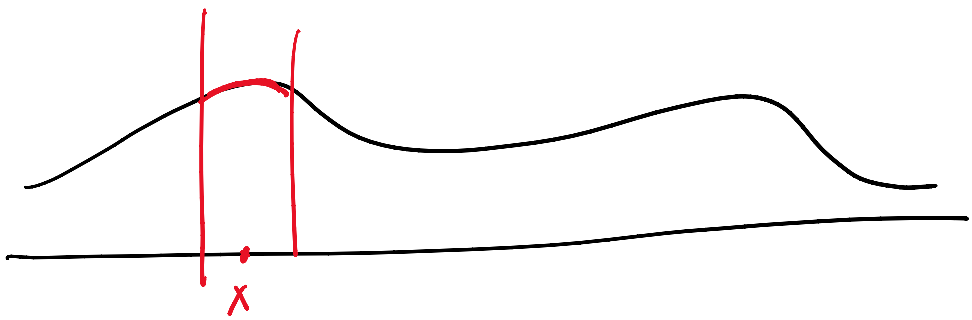
$$f * g(x) = \int_{\mathbb{R}^n} f(y) g(x-y)\, dy$$

e.g. $g = [-1, 1]$

$\mathbb{R}^n$ e.g. $g = [-1, 1]$

$$f*g(x) = \int_{\mathbb{R}} f(y)\, 1_{\{x-y\in[-1,1]\}}\, dy$$

$$= \int f(y)\, 1_{x\in[y-1,y+1]}\, dy$$



running sum/average.

Thm [Brunn-Minkowski] $A, B \subseteq \mathbb{R}^n$

$$A+B = \{x+y : x\in A,\ y\in B\}$$

$$\mathrm{vol}(A+B)^{\frac{1}{n}} \geq \mathrm{vol}(A)^{\frac{1}{n}} + \mathrm{vol}(B)^{\frac{1}{n}}.$$

(note: $\Rightarrow \mathrm{vol}(\lambda A+(1-\lambda)B)^{\frac{1}{n}} \geq \lambda\, \mathrm{vol}(A)^{\frac{1}{n}} + (1-\lambda)\,\mathrm{vol}(B)^{\frac{1}{n}}$)

Thm [Prékopa-Leindler]. $f, g, h : \mathbb{R}^n \longrightarrow \mathbb{R}_+$ $\lambda\in[0,1]$

$$h(\lambda x + (1-\lambda)y) \geq f(x)^{\lambda} g(x)^{1-\lambda}$$

$$\Rightarrow \int h \geq \left(\int f\right)^{\lambda} \left(\int g\right)^{1-\lambda}$$

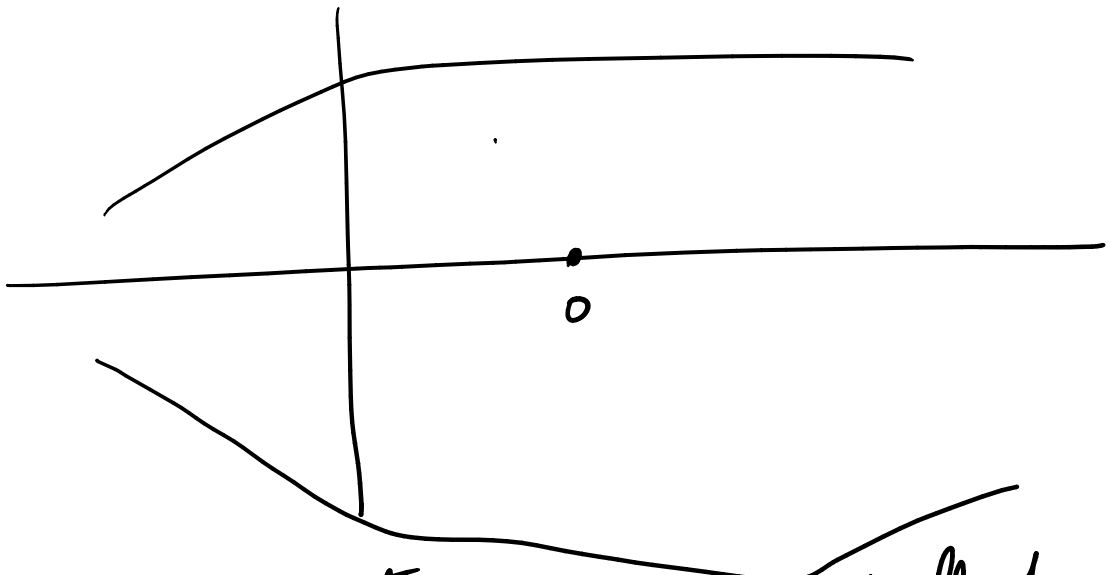
$$\Rightarrow \quad \int h \geq \left(\int f\right)^{\lambda} \left(\int g\right)^{1-\lambda}.$$

Back to center of Gravity.

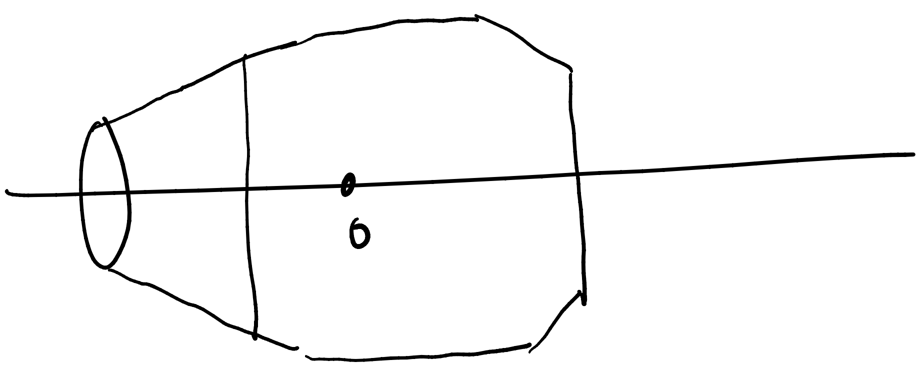
<u>Thm</u>. K. $\mathbb{E}_K(x) = 0$.

$$\mathrm{Vol}(K \cap \{x_1 \leq 0\}) \geq \frac{1}{e} \mathrm{Vol}(K).$$

<u>Pf</u>.



Replace each cross-section $A(x_1 = t)$ with a ball of same $(n-1)$-dim volume.



<u>Claim</u>: radius $r(t)$ is a <u>concave</u> function

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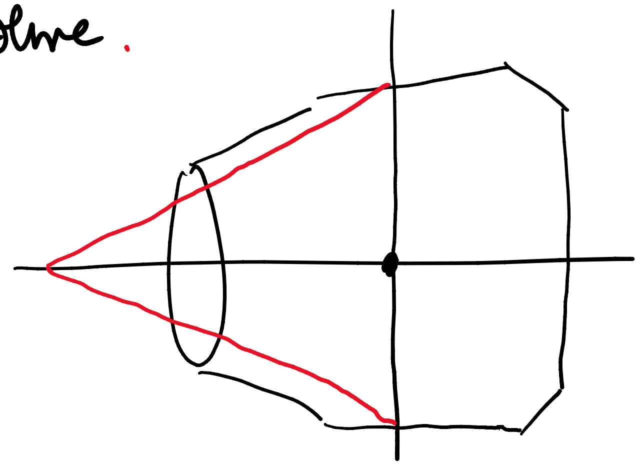
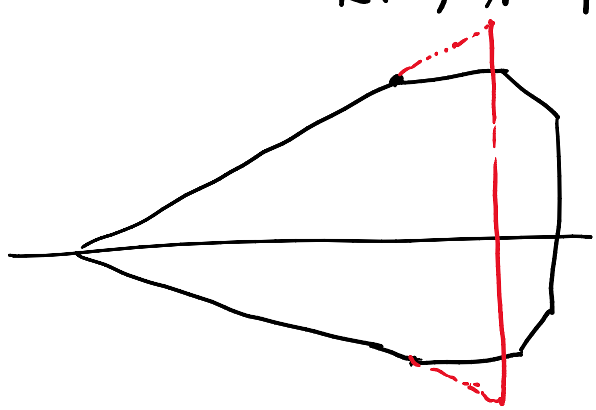
Pf. Apply B-M.

- Next, replace $x_1 \le 0$ part of K with a cone of same volume.



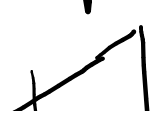
- moves CoG to the left, reducing volume of left halfspace.

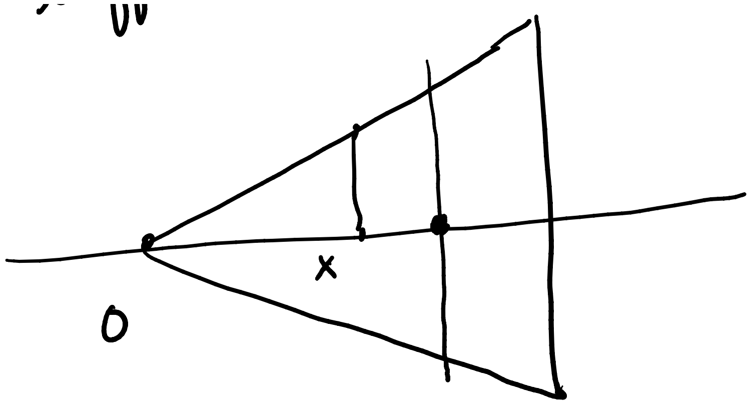
- Do the same on the RHS, keeping same cone.



- moves CoG $\leftarrow$.

So, suffices to prove for a cone!





$A(x) \propto x^{n-1}$. total volume $= \int_0^1 x^{n-1} = \frac{1}{n}$

$$\text{CoG} = \frac{\int_0^1 x \cdot x^{n-1}}{\frac{1}{n}} = \frac{\frac{1}{n+1}}{\frac{1}{n}} = \frac{n}{n+1}.$$

$$\frac{\text{volume}(K \cap H)}{\text{volume}(K)} = \frac{\int_0^{\frac{n}{n+1}} x^{n-1}}{\frac{1}{n}} = \left(\frac{n}{n+1}\right)^n = \left(1 - \frac{1}{n+1}\right)^n \geq \frac{1}{e}.$$

Q2. How to compute CoG?

Hard! #P-hard even for a polytope.

But $z = \mathbb{E}_K(x)$ is the average.

How about a random sample?

How about a random sample?

$$\begin{cases} E_{k+1} = E_k \cap \{a^T x \le a^T x^k\} \\ x^{k+1} = \frac{1}{m}\sum_{i=1}^{m} Y^i \quad Y^i \sim \text{uniform from } E_k. \end{cases}$$

<u>Thm 1</u>: X = avg of m random points from K.

H = Halfspace containing X.

$$\mathbb{E}(\mathrm{Vol}(K \cap H)) \geq \left(\frac{1}{e} - \sqrt{\frac{n}{m}}\right) \cdot \mathrm{Vol}(K).$$

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based on

<u>Thm 2</u>. [Robert Grunbaum]. Isotropic K. $\mathbb{E}_K(x) = 0$

$\mathbb{E}_K(xx^T) = I$.

H halfspace within distance t of 0.

Then $\mathrm{Vol}(K \cap H) \geq \left(\frac{1}{e} - t\right) \cdot \mathrm{Vol}(K)$.

Lemma $f: \mathbb{R} \to \mathbb{R}_+$, logconcave, isotropic.

$$\int f = 1 \quad \int x f(x) = 0 \quad \int x^2 f(x) = 1.$$

$$\Rightarrow \quad \max f \leq 1.$$

Pf (2) project to 1-d. normal of H.

Apply Lemma.

Pf. of 1.

$$\mathbb{E}(X) = 0$$

$$\mathbb{E}(\|X\|^2) = \frac{1}{m} \mathbb{E}(\|Y^i\|^2)$$

$$= \frac{n}{m}.$$

$$\mathbb{E}(\|X\|) \leq \sqrt{\frac{n}{m}}$$

Apply Thm 2 with $t = \sqrt{\frac{n}{m}}$.

So using $O(n)$, say $10n$ points suffices to get a constant-factor decrease in volume.