Central Path & IPM

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Yulu

Central Path

$$\min c^T x \quad Ax = b \quad x \geq 0 \qquad (P)$$

$$\min c^T x - t\sum_i \ln x_i \quad Ax = b \quad x \geq 0 \qquad (P_t)$$

$$\max b^T y \quad A^T y + s = c \quad s \geq 0$$

$t \to 0$

Lemma. $OPT(P_t)$ is a solution to:
(P_t has an interior pt)

$$xs = t \qquad (x_i s_i = t)$$
$$Ax = b$$
$$A^T y + s = c$$
$$s, x \geq 0$$

Maintain x^t, y^t, s^t

$(x^t, y^t, s^t) = C_t$

$$c^T x - b^T y = c^T x - x^T A^T y = x^T s \quad (= tn)$$

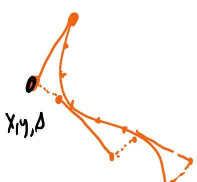
① Initial solution C_1

② step: Use C_t to find $C_{(1-h)t}$

③ Stopping Criterion: $t \leq \frac{\delta}{n}$

Given x, y, s find $\delta_x, \delta_y, \delta_s$

$t \Leftarrow (1-h)t$



x, y, s

$$\begin{cases} (x+\delta_x)(s+\delta_s) = t - xs \\ A\delta_x = 0 \\ A^T\delta_y + \delta_s = 0 \end{cases}$$

$$\boxed{\begin{aligned} xs &\approx t \\ Ax &= b \\ A^T y + s &= c \\ x, s &\geq 0 \end{aligned}}$$

✓

$$\begin{pmatrix} 0 & A^T & I \\ A & 0 & 0 \\ S & 0 & X \end{pmatrix} \begin{pmatrix} \delta_x \\ \delta_y \\ \delta_s \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ t - xs \end{pmatrix}$$

$S = \mathrm{Diag}(s) \quad X = \mathrm{Diag}(x) \qquad r = t - xs$

$S = \mathrm{diag}(s)$ ...

$$\delta_s = -A^T \delta_y$$

$$\delta_x = S^{-1}(r - X\delta_s) = S^{-1}(r + XA^T\delta_y)$$

$$A\delta_x = 0 \qquad AS^{-1}r + AS^{-1}XA^T\delta_y = 0$$

$$\delta_y = -(AS^{-1}XA^T)^{-1}AS^{-1}r$$

$$\delta_s = A^T(AS^{-1}XA^T)^{-1}AS^{-1}r$$

$$X\delta_s = \underbrace{XA^T(AS^{-1}XA^T)^{-1}AS^{-1}}_{Q}r = Qr$$

$$S\delta_x = r - Qr = (I-Q)r$$

Lemma. $x' = x + \delta_x$, $\boxed{y' = y + \delta_y}$, $s' = s + \delta_s$ are feasible, $x', s' \geq 0$, assuming

$$\sum_i (x_i s_i - t)^2 \leq \varepsilon^2 t^2 \text{ for some } \varepsilon < \tfrac{1}{2}. \qquad \|r\|_2^2 \leq \varepsilon^2 t^2$$

Pf. Let $P = S^{-\frac{1}{2}}X^{\frac{1}{2}}A^T(AS^{-1}XA^T)^{-1}AX^{\frac{1}{2}}S^{-\frac{1}{2}}$

P is a projection matrix

$B^T(BB^T)^{-1}B$ symmetric

$$P^2 = P \qquad \lambda_i^2 = \lambda_i \in \{0,1\}$$

$$B^T\cancel{(BB^T)^{-1}}\cancel{BB^T}(BB^T)^{-1}B$$

$\Rightarrow \forall i \quad x_i s_i \geq (1-\varepsilon)t$

$$\|X^{-1}\delta_x\|_\infty < 1 \Rightarrow |\delta_{x,i}| < x_i$$

$$\therefore x_i + \delta_{x,i} > 0$$

$$\|X^{-1}\delta_x\|_2 = \|X^{-1}S^{-1}(I-Q)r\|$$

$$= \|S^{-\frac{1}{2}}X^{-\frac{1}{2}}(I-P)X^{-\frac{1}{2}}S^{-\frac{1}{2}}r\|$$

$$= \| S^{\frac{1}{2}} X^{\frac{1}{2}} (I - P) X \cdots \|$$

$$\leq \frac{1}{\sqrt{(1-\varepsilon)t}} \cdot \| (I-P) X^{-\frac{1}{2}} S^{-\frac{1}{2}} r \|_2$$

$$\leq \frac{1}{\sqrt{(1-\varepsilon)t}} \| X^{-\frac{1}{2}} S^{-\frac{1}{2}} r \|_2$$

$$\leq \frac{1}{(1-\varepsilon)t} \| r \|_2 \leq \frac{\varepsilon t}{(1-\varepsilon)t} \leq \frac{\varepsilon}{1-\varepsilon}.$$

$\varepsilon < \frac{1}{2} \implies \| X^{-1} \delta_x \|_\infty \leq \| X^{-1} \delta_x \|_2 < 1.$

<u>Illy</u> $\| S^{-1} \delta_s \|_\infty < 1.$ $\quad x', s' > 0$ ✓.

<u>Lema 2.</u> Assume $\sum_i (x_i s_i - t)^2 \leq \varepsilon^2 t^2$, $\varepsilon < \frac{1}{4}$

$$\implies \sum_i (x_i' s_i' - t)^2 \leq (\varepsilon^4 + 16 \varepsilon^5) t^2$$

- Each step (solving Linear system for $\delta_x, \delta_y, \delta_s$) brings us closer to the central path.

- $t \leftarrow (1-h)t$.

<u>Pf.</u> $\forall i$ $x_i \delta_{s,i} + s_i \delta_{x,i} = t - x_i s_i$

$$\sum_i (x_i' s_i' - t)^2 = \sum_i (x_i s_i + \underline{x_i \delta_{s,i} + s_i \delta_{x,i}} + \delta_{x,i} \delta_{s,i} - t)^2$$

$$\ldots \varepsilon^2 t^2 \leq (\delta_{x,i})^2 (\delta_{s,i})^2$$

$$= \sum_i \delta_{x,i}^2 \delta_{s,i}^2 \le (1+\varepsilon)^2 t^2 \sum_i \left(\frac{\delta_{x,i}}{x_i}\right)^2 \left(\frac{\delta_{s,i}}{s_i}\right)^2$$

$x_i s_i \le (1+\varepsilon) t$

$$\le (1+\varepsilon)^2 t^2 \, \|X^{-1}\delta_x\|_4^2 \, \|S^{-1}\delta_s\|_4^2$$

$$\le (1+\varepsilon)^2 t^2 \, \|X^{-1}\delta_x\|_2^2 \, \|S^{-1}\delta_s\|_2^2$$

$$\le (1+\varepsilon)^2 t^2 \left(\frac{\varepsilon}{1-\varepsilon}\right)^4$$

$$\le \left(\varepsilon^4 + O(\varepsilon^5)\right) t^2 .$$

Thm. LP can be solved to within δ error using $O\left(\sqrt{n} \log \frac{1}{\delta}\right)$ iterations, where each iteration solves a single linear system.

$t \to \left(1 - \frac{1}{10\sqrt{n}}\right) t$

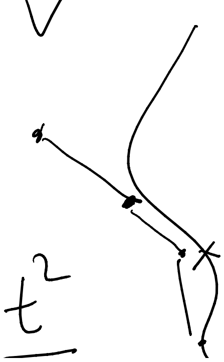
$\downarrow \delta$

Pf. $\Phi = \sum_i (x_i s_i - t)^2$ $\qquad \varepsilon < \frac{1}{4}$.

Maintain $\Phi \le \frac{t^2}{16}$.

(1) Reduce $\Phi \longrightarrow \le \frac{t^2}{50}$. ✓

(2) decrease t



$\sum (x_i s_i - t(1-h))^2 \le \frac{t^2}{\ldots}$

$$\sum_i \left(x_i s_i - t(1-h)\right)^2 \le \frac{t^2}{16}$$

largest h.

$$\sum_{i=1}^{n} \left(x_i s_i - t + th\right)^2$$

$$\le 2\sum_i (x_i s_i - t)^2 + 2t^2h^2 n$$

$$\le 2\cdot\frac{t^2}{50} + \frac{2t^2\cdot \cancel{n}}{100\cdot\cancel{n}} \le \frac{t^2}{25} \le \boxed{\frac{t^2(1-h)^2}{16}}$$

$$\boxed{h = \frac{1}{10\sqrt{n}}}$$

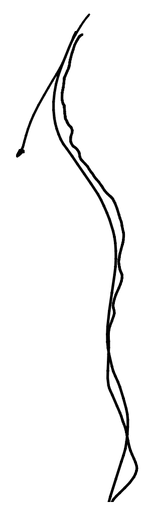
$$h = \omega\left(\frac{1}{\sqrt{n}}\right)$$

$$\longrightarrow \boxed{Ax = b} \longrightarrow x$$

$$t \longrightarrow \boxed{\frac{dx_t}{dt} = f(A, b, x_t)} \longrightarrow x_t$$

$$\boxed{\begin{aligned} S_t \frac{dx_t}{dt} + X_t \frac{dx_s}{dt} &= 1 \\ A\frac{dx_t}{dt} &= 0 \end{aligned}}$$

$t = 1$



$$\frac{\quad}{dt}$$

$$\boxed{A^T \frac{dy_t}{dt} + \frac{ds_t}{dt} = 0}$$

$\downarrow$

$0.$

$$S_t \frac{dx_t}{dt} = (I - Q_t) \cdot \mathbb{1}, \quad X_t \frac{ds_t}{dt} = Q_t \cdot \mathbb{1}.$$

$$Q_t = X_t A^T (A S_t^{-1} X_t A^T)^{-1} A S_t^{-1}$$

$$\boxed{x_t s_t = t}$$

$$Q_t = X_t A^T (A X_t^2 A^T)^{-1} A X_t$$

$$= S_t^{-1} A^T (A S_t^{-2} A^T)^{-1} A S_t^{-1}$$

$$X_t^{-1} \frac{dx_t}{dt} = \frac{1}{t} (I - P_t) 1.$$

$$S_t^{-1} \frac{ds_t}{dt} = \frac{1}{t} \cdot P_t \cdot \mathbb{1}.$$

$$\frac{d \ln x_t}{dt} = \frac{1}{t} (I - P_t) \cdot 1$$

$$\boxed{\begin{aligned} \frac{d \ln x_t}{d \ln t} &= (I - P_t)\mathbb{1} \\ \frac{d \ln s_t}{d \ln t} &= P_t \mathbb{1}. \end{aligned}}$$

$$\begin{aligned} &\|P_t \mathbb{1}\|_\infty \\ &\le \|P_t \mathbb{1}\|_2 \\ &\le \sqrt{n}. \end{aligned}$$

max change is $\left(1 \pm \frac{1}{\sqrt{n}}\right)$