Pumping

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We have seen how to show there exist languages that are undecidable, i.e., $x \in L$? cannot be decided by any TM.

What about explicit ones?

Consider

$L_A = \{ \langle M, x \rangle : \text{TM } M \text{ accepts string } x \}$

<u>Thm</u>. L_A is undecidable.

<u>Pf.</u> Suppose not, i.e. $\exists$ TM D that can decide if a given string is in L_A.

Consider the following TM $\hat{D}$:

Given $\langle M \rangle$,

run D on $\langle M, \langle M \rangle \rangle$

if D says ACCEPT, then reject.

if D says REJECT, then accept.

What will $\hat{D}$ do on input $\langle \hat{D} \rangle$?

$\hat{D}$ will accept iff D rejects $\langle \hat{D}, \langle \hat{D} \rangle \rangle$

i.e. iff $\langle \hat{D} \rangle \notin L_{\hat{D}}$.

contradiction!

$\hat{D}$, D do not exist!

$L_{HALT} = \{ \langle M, x \rangle : M \text{ halts on input } x \}$

Thm. L_{HALT} is undecidable.

Pf. Suppose $\exists$ TM H to decide L_{HALT}.

Then consider the following TM A:

On input $\langle M, x \rangle$:

- run H on $\langle M, x \rangle$
- if H rejects, reject
- else (H accepts),

  run M on x

  - accept if M accepts x
  - reject if M rejects x.

- accept if

- reject if M rejects x.

TM A decides L_A ! But that is impossible ! ∎

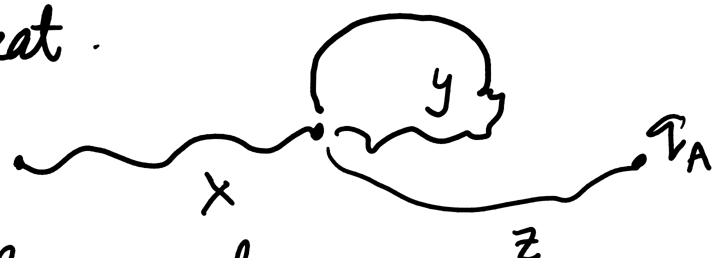
Back to finite automata.

$\{0^n 1^n\}$ $\exists$ DFA ?

$\{1^n : n$ is a power of $2\}$

Suppose $\exists$ DFA with p states. For $n \geq p$,

a state must repeat.



but then xy^2z is also accepted

xy^3z

⋮

1^{a+b+c}

$a+b+c = 2^k$

$a+2b+c = 2^k + b \geq 2^{k+1}$

So $a = c = 0$

but then $3b, 4b \ldots$ accepted.

but then 3b, 4b ... accepted.

Thm. L is a regular language, $\exists\, p$ s.t.
$\forall\, s$ of length at least p, $s = xyz$ s.t.

1. $\forall\, i \geq 0 \quad xy^iz \in L$
2. $|y| > 0$
3. $|xy| \leq p$.

Pf. L is regular $\Rightarrow$ $\exists$ DFA for L.

set $p = |Q|$.

Take any $s \in L$, $|s| = n \geq p$

sequence of states

$q_0, q_1, \ldots \; q_i \; q_{i+1}, \ldots \; q_j \; q_{j+1} \; \cdots \; q_n$

Since $n+1 > p$, $\exists$ repeated state q_i

q_i is the first Say $q_i = q_j$

$q_0 \, q_1 \ldots \quad q_i \quad q_{i+1} \; \ldots \; q_{j-1} \; q_j = q_i \; q_{j+1} \ldots q_n$

$s_1 \; \ldots \; s_i \quad s_{i+1} \; \ldots \; s_j \quad s_{j+1} \quad s_n$

$$\underbrace{s_1 \quad \dots \quad s_i}_{X} \quad \underbrace{s_{i+1} \quad \dots \quad s_j}_{Y} \quad \underbrace{s_{j+1} \quad s_n}_{Z}$$

Now consider XZ, XY^2Z, XY^iZ

all $\in L$.

①

② $\exists$ at least one symbol between q_i and q_j, so $|Y| > 0$

③ $|XY| \le p$. since q_j is the first repetition.

<u>Ex.</u> $\{0^n 1^n\}$ is not regular.

<u>Pf.</u> $0^n 1^n \in L$. $\quad n \ge p$ (# states)

$0^n 1^n = XYZ$

Since $|XY| \le p$, $Y = 0^i$ for some $i > 0$

So $0^{n+i} 1^n \in L$. (*) !