CS 4510: Automata and Complexity **Fall 2019**

Lecture 9: Probabilistic Finite Automata II

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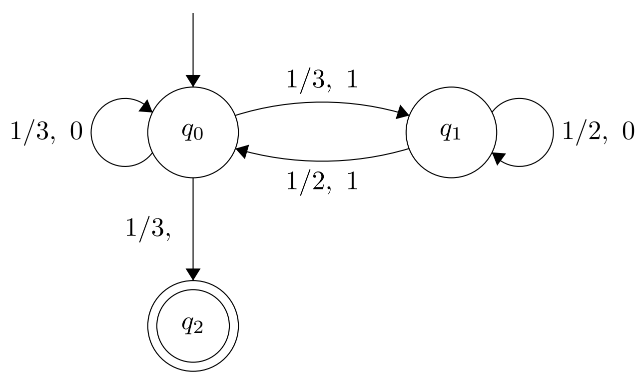
Definition of Probabilistic FA

The formal definition of PFA is given below:

- Set of states **Q**
- Transition matrix $\mathbf{P} \geqslant 0$ whose row sums = 1(Each transition can have some output letter)
- Starting distribution $\pi^{(0)}$
- Set of end states

PFA Example

Design a PFA that only output binary strings with an even $\#1's$.



Notice that PFA might not have an end state.

Definition 9.1 ***Support graph*** *G=(V,E) (V is the set of vertices and E is the set of edges) of transition matrix is a graph in which each edge is a transition with positive probability value.*

Definition 9.2 *A matrix is **aperiodic** $\Longleftrightarrow$ GCD(all lengths of directed cycles in its support graph) = 1.*

Lemma 9.3 *If a matrix $\boldsymbol{P}$ is primitive, or $\boldsymbol{P}$ is irreducible and aperiodic, then its distribution $\pi^{(t)}$ converges to a stationary distribution (or steady state) π, i.e. $\pi^{(t)} \longrightarrow \pi$.*

A Special PFA

Suppose G = (Q,E) is the support graph of a PFA. Let the degree of Q_i (or the number of transitions at the state Q_i represents) be d_i.

Let $P_{ij} = \frac{1}{d_i}$, which means all transitions at state Q_i are equally likely. Then we have

$$\mathbf{P} \cdot \mathbf{1} = \mathbf{P} \cdot \begin{pmatrix} 1 \\ \vdots \\ \vdots \\ \vdots \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ \vdots \\ \vdots \\ \vdots \\ 1 \end{pmatrix} = \mathbf{1}$$

which means vector **1** is a eigenvector of this matrix and its eigenvalue is 1.

Question: Does this matrix have a stationary distribution? If so, what is π?

Lemma 9.4 $\pi = \frac{1}{\sum_i d_i} \begin{pmatrix} d_1 \\ d_2 \\ \vdots \\ d_n \end{pmatrix}$ *is stationary distribution for* $\boldsymbol{P}$.

Proof: The probability of being at state Q_i at some step is $(P^T \pi)_i$.

$$(P^T \pi)_i = \sum_j \pi_j \cdot P_{ji} = \sum_{j:(j,i) \in E} \frac{\pi_j}{d_j} = \frac{d_i}{\sum d_i} = \pi_i$$

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Question: What is the probability of going from i to j in the steady state π?

$$\pi_i \cdot P_{ij} = \frac{d_i}{\sum d_i} \cdot \frac{1}{d_i} = \frac{1}{\sum d_i} = \frac{1}{2m} \ (m = \#edges)$$

Each transition is equally likely in steady state.

Simple Random Walk

Suppose we have a graph that is connected and aperiodic (or abipartite). There are three questions that we want to ask about this graph:

1. Access (hitting) time H(i,j): starting from i, how long does it take to get to j?

$$H(i,j) = \mathbb{E}(\#\text{steps to go from } i \text{ to } j)$$

2. Cover time C(i): starting from i, how long does it take to visit every edge?

$$C(i) = \mathbb{E}(\#\text{steps to visit all vertices starting at } i)$$

3. Mixing rate: rate at which $\pi^{(t)}$ approach π (or $\pi^{(t)}$ converge). Defined as

$$\mu = \limsup_{t \longrightarrow \inf} \max_{i,j} |P_{ij}^{(t)} - \pi_i|^{1/t}$$

Example 1: Path

Consider the simplest graph, a path. Assume the path has $n+1$ vertices and n edge. Each edge has a label in $\{0, 1, 2, \ldots, n\}$.

For this graph, it's steady state is

$$\pi(x) = \begin{cases} \frac{1}{n}, & x = 1, 2, 3, \ldots, n-1. \\ \frac{1}{2n}, & x = 0, n. \end{cases} \tag{9.1}$$

Consider the hitting time $H(0, n)$:
$H(i-1, i) = 2i - 1$
$H(i, j) = H(i, j-1) + H(j-1, j) = H(i, j-1) + (2j-1)$
$H(i, n) = (2n-1) + (2n-3) + (2n-5) + \cdots + (2i+1) = \sum_{j=1}^{n} 2j - 1 - \sum_{j=1}^{i} 2j - 1 = n^2 - i^2$
$H(0, n) = n^2$

Example 2: Complete Graph

Complete graph is a graph in which each pair of vertices is connected with one edge. Suppose the number of vertices is n, then the number of vertices is $n(n-1)/2$. In a complete graph, $P_{ij} = \frac{1}{n-1}$ and stationary distribution $\pi(i) = \frac{1}{n}$ $(1 \leqslant i, j \leqslant n)$. The hitting time $H(i, j) = n - 1$.

Now let's consider the cover time of this graph:

Suppose t_i is the number of steps when visiting i different vertices for the first time, the following relation holds:

$$0 \leqslant t_1 \leqslant t_3 \leqslant \cdots \leqslant t_n$$

And the probability of visiting a new vertex after t_i is $\frac{n-i}{n-1}$ and $\mathbb{E}(t_{i+1} - t_i) = \frac{n-1}{n-i}$

$$\mathbb{E}(t_n) = \sum_{i=0}^{n-1} \mathbb{E}(t_{i+1} - t_i) = \sum_{i=0}^{n-1} \frac{n-1}{n-i} = (n-1) \sum_{i=1}^{n} \frac{1}{i} = n \ln n$$