PFA : Markov Chains

Wednesday, September 25, 2019 5:48 AM

- GAME

Last Time we introduced PFAs.

- states Q
- Transition matrix $P \geq 0$, row sums $= 1$.
- starting distribution $\pi^{(0)}$.
- End states.

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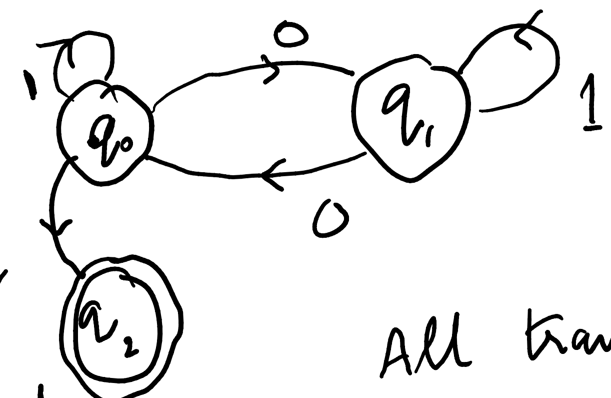
e.g. PFA that only outputs strings with an even #1's:



All transitions have > 0 probability.

no output.

PFA might not have an end state.

If P is primitive, or P is irreducible + aperiodic, ... directed cycles in

If P is primitive, so P is irreducible + [illegible]

[Aperiodic : GCD of all lengths of directed cycles in P is 1]

then $\pi^{(t)} \longrightarrow \pi$ unique.

We consider a simple and natural setting today. $G = (Q, E)$ be the support of the PFA.

Let the degree of Q_i be d_i

Let $P_{ij} = \frac{1}{d_i}$. $\left(P_{ji} = \frac{1}{d_j}\right)$

i.e. we pick a transition at random.

$$P = \begin{pmatrix} \frac{1}{d_1} & \frac{1}{d_1} & \cdots & \\ & & \frac{1}{d_j} & \\ & & & \end{pmatrix}$$

Note $P \mathbb{1} = \mathbb{1}$

$$\left(P^T \pi^{(t)}\right)_i = \sum_j P_{ji} \pi_j^{(t)} = \sum_{\substack{j : (j,i) \\ \in E}} \frac{\pi_j^{(t)}}{d_j}$$

<u>Lemma</u>. $\pi = \frac{1}{\sum d_i} \begin{pmatrix} d_1 \\ \vdots \\ d \end{pmatrix}$ is stationary for P.

$$\pi = \frac{1}{\sum d_i}\begin{pmatrix} \vdots \\ d_n \end{pmatrix}$$

<u>Pf.</u> $$(P^T\pi)_i = \sum_{j:(j,i)\in E} \frac{\pi_j}{d_j} = \frac{d_i}{\sum d_i} = \pi_i .$$

<u>Lemma.</u> If G is connected and non-bipartite then π is the unique stationary distribution and $\pi^{(t)} \longrightarrow \pi$.

Q. What is the probability of going from i to j in the steady state π?

$$\pi_i \cdot P_{ij} = \frac{d_i}{\sum d_i} \cdot \frac{1}{d_i} = \frac{1}{\sum d_i} = \frac{1}{2m}.$$

"each transition is equally likely".

What we have is a simple random walk on a graph. This is an object of much study and has many applications.

<u>Parameters:</u> Access time $H(i,j) = E(\#\text{ steps to go from } i \text{ to } j)$

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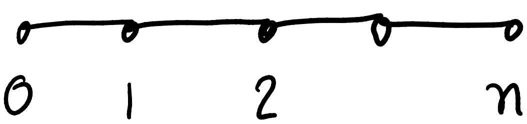
Cover time $C(i) = E($# steps to visit all vertices starting at $i)$

Mixing time $\mu = \sup_{t \to \infty} \max_{i,j} \left| p_{ij}^{(t)} - \pi_j \right|^{1/t}$.

"rate at which $\pi^{(t)} \to \pi$".

Examples.

1. Path



$H(0, n)$?

$$\pi(0) = \frac{1}{2n} \qquad \pi(n) = \frac{1}{2n}$$

$$\pi(i) = \frac{2}{2n} = \frac{1}{n} \qquad i = 1, 2, \ldots n-1$$

— break ——

$H(i-1, i) = 2i - 1$.

$H(i, j) = H(i, j-1) + 2j - 1$

$$H(i, n) = 2n-1 + 2n-3 + 2n-5 \ldots 2i+1$$

$$= \sum_{i=1}^{n} 2j-1 - \sum_{i=1}^{i} 2j-1$$

$$= \sum_{j=1} 2j-1 \quad - \sum_{j=1} 2j-1$$

$$= n^2 - i^2$$

$$\boxed{H(0,n) = n^2}$$

2. Complete graph.

$$P_{ij} = \frac{1}{n-1} \qquad \pi(i) = \frac{1}{n}.$$

$$H(i,j) = n-1.$$

What about Cover time?

— thinking break —

$$0 = t_1 \leq t_2 \ \ldots \ t_i \quad \leq t_n$$

first time visiting i vertices

$$\Pr(\text{new vertex after } t_i) = \frac{n-i}{n-1}$$

$$\mathbb{E}(t_{i+1} - t_i) = \frac{n-1}{n-i}$$

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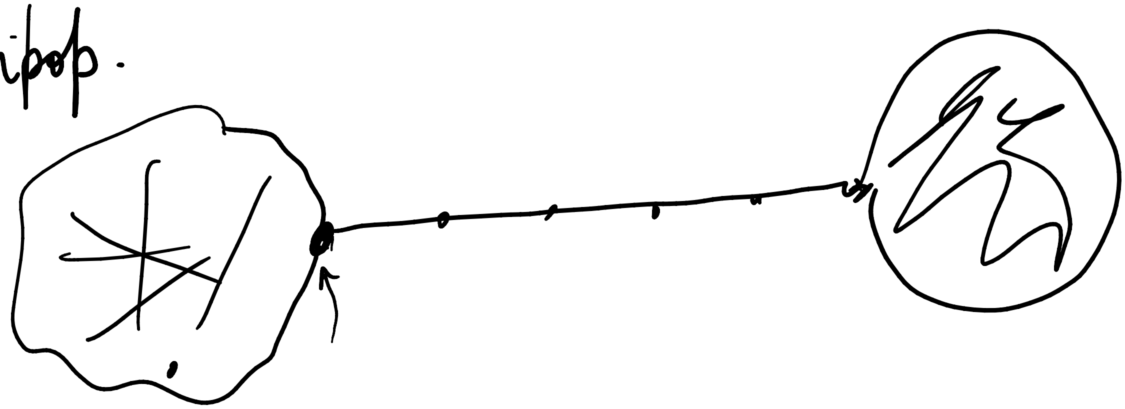
$$\mathbb{E}(t_n) = \sum_{i=0}^{n-1} \mathbb{E}(t_{i+1} - t_i) = \sum_{i=0}^{n-1} \frac{n-1}{n-i} = (n-1) \sum_{i=1}^{n} \frac{1}{i}$$

$$\simeq n \ln n.$$

3. Lollipop.



$\Theta(n^3)$.

<u>Thm</u>. The comm time for any connected undirected graph is $O(n^3)$.

<u>Mixing time</u>.

$P^T \pi = \pi$ eigenvalue 1

$\forall v \neq \pi \quad P^T v = \lambda v \quad \lambda < 1$

If P is primitive, $|\lambda| < 1$.

$$\chi^2(\pi^{(t+1)}, \pi) \le \lambda \cdot \chi^2(\pi^{(t)}, \pi).$$

So distance drops by factor λ in each step.

$$D = \begin{pmatrix} d_1 & & 0 \\ & \ddots & \\ 0 & & d_n \end{pmatrix}$$

$$P = D^{-1} A$$

adjacency matrix

$A_{ij} = 1 \quad (i,j) \in E$

consider

$$Q = D^{\frac{1}{2}} P D^{-\frac{1}{2}}$$
$$= D^{-\frac{1}{2}} A D^{-\frac{1}{2}}$$

<u>Lemma</u>. Q has the same eigenvalues as P.

<u>Pf</u>. Suppose $Pv = \lambda v$.

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$$Q D^{\frac{1}{2}} v = D^{\frac{1}{2}} P D^{-\frac{1}{2}} D^{\frac{1}{2}} v$$
$$= D^{\frac{1}{2}} P v$$
$$= \lambda D^{\frac{1}{2}} v.$$

$D^{\frac{1}{2}} v$ is an eigenvector of Q with same eigenvalue.

Thm [Spectral] For any real, symmetric $n \times n$ Q,

$$Q = \sum_{i=1}^{n} \lambda_i v_i v_i^T \qquad \lambda_1 \geq \lambda_2 \ldots \geq \lambda_n$$

real

$\{v_i\}$ are orthonormal

$\|v_i\| = 1$

$v_i^T v_j = 0 \quad \forall i \neq j$

Cor. $$Q^{(t)} = \sum^{n} \lambda_i^t v_i v_i^T$$

Cor. $Q^{(t)} = \sum_{i=1} \lambda_i \, v_i v_i$

Th $P^{(t)}_{ij} = \pi_j + \sum_{\ell=2}^{n} \lambda_\ell^t \, v_\ell v_\ell^T \sqrt{\frac{d_j}{d_i}}$

Pf. P, Q. $\lambda_1 = 1$

$$P^{(t)} = \left(D^{-\frac{1}{2}} Q \, D^{\frac{1}{2}}\right)^t$$

$$= D^{-\frac{1}{2}} Q^{(t)} D^{\frac{1}{2}}$$

$$= \sum_{\ell=1}^{n} \lambda_\ell^t \, D^{-\frac{1}{2}} v v^T D^{\frac{1}{2}}$$

$$= \pi \mathbb{1}^T + \sum_{\ell=2}^{n} \lambda_\ell^t \, D^{-\frac{1}{2}} v v^T D^{\frac{1}{2}}$$

$$P^t_{ij} = \pi_j + \sum_{\ell=2}^{n} \lambda_\ell^t \, v_{\ell i} v_{\ell j} \sqrt{\frac{d_j}{d_i}}$$

$$\therefore \quad \left| \pi_j^{(t)} - \pi_j \right| \leq \max_{2 \leq \ell \leq n} |\lambda_\ell|^t \sqrt{\frac{d_j}{d_i}}$$

starting at i.

Surprising fact:

Mixing time can be $O(\log n)$ for an n-node graph!