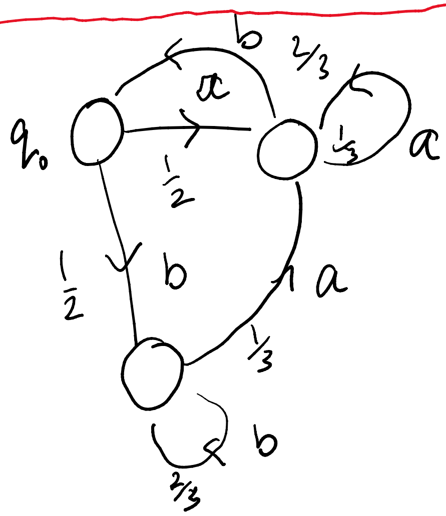
Probabilistic Finite Automata

Monday, September 23, 2019 6:04 AM

GAME!!! Discuss exam



ab	$\frac{1}{2} \cdot \frac{2}{3}$
aa	$\frac{1}{2} \cdot \frac{1}{3}$
ba	$\frac{1}{2} \cdot \frac{1}{3}$
bb	$\frac{1}{2} \cdot \frac{2}{3}$

At each time t ($t = 0, 1, 2, \ldots\ldots$), starting at q_0 at $t=0$, the current state is not fixed.

It has a distribution

$$\Pr(q^{(t)} = q) = \pi^{(t)}$$

What is $\pi^{(t+1)}$?

$$\pi^{(0)} = \begin{array}{cc} \begin{bmatrix} 1 \\ 0 \\ \vdots \\ \vdots \\ 0 \end{bmatrix} & \begin{array}{c} q_0 \\ \vdots \\ \vdots \\ \vdots \\ q_{n-1} \end{array} \end{array}$$

Transition matrix P

P_{ij} = Prob of going to j from i

$$\begin{array}{c|c|c|c|} & q_0 & q_1 & q_2 \\ \hline q_0 & 0 & \frac{1}{2} & \frac{1}{2} \end{array} = 1$$

Notice (t)

	q_0	q_1	q_2	
q_0	0	$\frac{1}{2}$	$\frac{1}{2}$	$=1$
q_1	$\frac{2}{3}$	$\frac{1}{3}$	0	$=1$
q_2	0	$\frac{1}{3}$	$\frac{2}{3}$	$=1$

Notice

$$\sum_i \pi_i^{t+1} = \sum_{i,j} P_{ji}\, \pi_j^{(t)}$$

$$= \sum_j \pi_j^{(t)} \cdot \sum_i P_{ji}$$

$$= \sum_j \pi_j^{(t)}$$

<u>Lemma.</u> $\pi^{(t+1)} = P^T \pi^{(t)}$.

$$\boxed{\text{Q. What is } \lim_{t \to \infty} \pi^{(t)} \text{ ?}}$$

$$\begin{pmatrix} 0 & \frac{2}{3} & 0 \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{2} & 0 & \frac{2}{3} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{1}{2} \\ \frac{1}{2} \end{pmatrix} \to \begin{pmatrix} \frac{1}{3} \\ \frac{1}{3} \\ \frac{1}{3} \end{pmatrix}$$

$$\downarrow$$

$$\begin{pmatrix} \frac{2}{9} \\ \frac{7}{18} \\ \frac{7}{18} \end{pmatrix}$$

$$\begin{pmatrix} \quad & \quad & \quad \end{pmatrix} \begin{pmatrix} a \\ b \\ b \end{pmatrix} = \begin{pmatrix} \frac{2}{3} b \\ \frac{a}{2} + \frac{2}{3} b \\ \frac{a}{2} + \frac{2}{3} b \end{pmatrix}$$

$$\begin{pmatrix} 1-2b \\ b \\ b \end{pmatrix} \qquad \begin{pmatrix} \frac{2}{3} b \\ \frac{1}{2} - \frac{1}{3} b \\ \frac{1}{2} - \frac{1}{3} b \end{pmatrix}$$

$\boxed{\text{fixed point}}$ $\quad 1-2b = \frac{2}{3} b \qquad b = \frac{3}{8}$

$\left(\frac{1}{\cdot}\right. \quad \left(1-2b\right. \quad \left(2b-\frac{3}{4}\right.$ $\quad \chi^2$ distance:

$$\begin{pmatrix} \frac{1}{4} \\ \frac{3}{8} \\ \frac{3}{8} \end{pmatrix} - \begin{pmatrix} 1-2b \\ b \\ b \end{pmatrix} = \begin{pmatrix} 2b-3/4 \\ \frac{3}{8}-b \\ \frac{3}{8}-b \end{pmatrix}$$

χ^2 distance:

$$\sum \left(\frac{p_i}{\pi_i} - 1\right)^2 \pi_i$$

vs $$\begin{pmatrix} \frac{1}{4} \\ 3/8 \\ 3/8 \end{pmatrix} - \begin{pmatrix} 2/3\, b \\ \frac{1}{2}-\frac{1}{3}b \\ \frac{1}{2}-\frac{1}{3}b \end{pmatrix} = \begin{pmatrix} \frac{1}{4}-2/3\, b \\ \frac{1}{3}b-\frac{1}{8} \\ \frac{1}{3}b-\frac{1}{8} \end{pmatrix}$$

$$\sum_i \frac{(p_i - \pi_i)^2}{\pi_i}$$

$$4.\left(2b-3/4\right)^2 + \frac{8}{3}.2.\left(\frac{3}{8}-b\right)^2$$

vs.

$$4.\left(\frac{1}{4}-\frac{2}{3}b\right)^2 + \frac{8}{3}.2.\left(\frac{1}{3}b-\frac{1}{8}\right)^2$$

$$\left(16b^2 + \frac{16}{3}b^2\right).\frac{1}{9} = \frac{16}{9}b^2 + \frac{16}{27}b^2$$

$$\left(4.\frac{9}{16} + \frac{16}{3}.\frac{9}{8^2}\right).\frac{1}{9} = 4.\frac{1}{4^2} + \frac{16}{3}.\frac{1}{8^2}$$

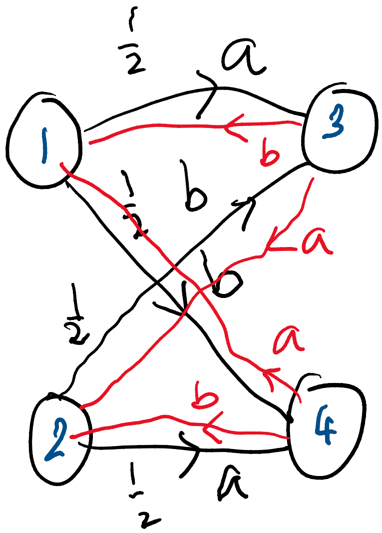
$$-4.\cancel{4}.b.\frac{3}{\cancel{4}} - \frac{\cancel{16}^2}{\cancel{3}}.\frac{\cancel{3}}{\cancel{8}}.2b$$

$$-16b \quad \text{vs} \quad -\frac{4}{3}.\frac{4}{3} \qquad -\cancel{4}.2.\frac{1}{\cancel{4}}.\frac{2}{3}.b \quad - \frac{\cancel{16}^2}{3}.\frac{1}{\cancel{8}}.\frac{2}{3}b$$

$$(a^2+b^2+2ab) \longrightarrow \frac{1}{9}(a^2+b^2+2ab) \longrightarrow 0 \;!!$$

$$\boxed{\text{Is this general?}}$$

Is this general?



$$P = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{pmatrix} & & \frac{1}{2} & \frac{1}{2} \\ & & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & & \\ \frac{1}{2} & \frac{1}{2} & & \end{pmatrix} \end{array}$$

$$P^T \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \frac{1}{2} \\ \frac{1}{2} \end{pmatrix} \rightarrow \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 0 \\ 0 \end{pmatrix} \rightarrow \begin{pmatrix} 0 \\ 0 \\ \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}$$

YIKES!

Consider
$$P = \left(\begin{array}{c|c} 0 & A \\ \hline B & 0 \end{array}\right)$$

$$P^2 = \begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix} \begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix} = \begin{pmatrix} AB & 0 \\ 0 & BA \end{pmatrix}$$

$$P^3 = \begin{pmatrix} 0 & ABA \\ BAB & 0 \end{pmatrix}$$

Nonnegative matrix P is <u>primitive</u>

if $\exists k$ s.t. $\forall i,j$ $(P)^k_{i,j} > 0$.

<u>Thm</u> [Perron] For primitive matrix P,

$$P^k \cdot x \to v \qquad x \geq 0,\ x \neq 0$$

(1) $v > 0$

(2) $Pv = \lambda v$

(3) $\forall u$ $Pu = \alpha u$, $\alpha < \lambda$.

Nonnegative matrix P is <u>irreducible</u>

if $\forall i,j$ $\exists K$: $(P)^K_{i,j} > 0$

Thm [FROBENIUS] For irreducible matrix P,
$\exists v$: (1), (2), (3) hold.

PRIMITIVE: $\Longleftrightarrow$ strongly connected support
and aperiodic.

Then $\pi^k \to \pi$ unique.