CS 4510: Automata and Complexity	Fall 2019

Lecture 17: P and NP

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17.1 P

We come to a central definition in complexity theory, P, which represents the class of languages that can be decided in polynomial time, in other words, there exists $k \in \mathbb{N}$ such that $L \in TIME(n^k)$, where $n = |x|$, length of input and k is fixed, independent of n.

$$P = \bigcup_k TIME(n^k)$$

17.1.1 Examples of classes in P

- $\{\langle G \rangle : G$ is a connected graph$\}$
- $\{\langle s_1, s_2, k \rangle :$ the edit distance between s_1 and s_2 is at most $k\}$

17.2 NP

We have another class of languages that are decidable in non-deterministic polynomial time. For L to be in NP, there exists a k and there exists a nondeterministic Turing machine, M such that for all x, $|x| = n$, if x is in L, M has an accepting path of length $\leq n^k$.

$$NP = \bigcup_k NTIME(n^k)$$

There exists a "short" proof of membership. x in L has a certificate of length $\leq n^k$.

17.2.1 Examples of classes in NP

- $\{\langle G \rangle : G$ is Hamiltonian$\}$ (there exists a path that visits each vertex in G exactly once)
- $\{\langle G, k \rangle : G$ has a clique of size $k\}$ (A clique is a subset of vertices in G such that every two distinct vertices are adjacent.)
- $\{\langle a_1, a_2, \ldots a_k \rangle :$ there exists a partition A_1, A_2 of these integers such that $\sum_{i \in A_1} a_i = \sum_{i \in A_2} a_i\}$

Clearly, P is a subset of NP.

17.3 Co-NP

A language L is in NP if and only if $\overline{L}$ is in $Co-NP$. $Co-NP$ is defined as a class of languages where there does not exists a nondeterministic Turing machine that accepts L in at most n^k steps.

$$\bigcup_k \{L : \text{ there does not exist NTM that accepts } L \text{ in} \leq n^k \text{ steps}\}$$

Both $x \in L$ and $x \notin \overline{L}$ has short proof.

17.3.1 Examples of classes in Co-NP

$$\{\langle G \rangle : G \text{ is } \underline{\text{not}} \text{ Hamiltonian}\}$$

$$\{\langle G, k \rangle : G \text{ does not have a clique of size } k\}$$

Theorem 17.1 *P is a subset of the intersection of NP and $Co - NP$*

One question we need to consider is: does $P = NP$? To answer this question, we'll see that a large class of problems such that solving any one of them will imply $P = NP$.

17.3.2 More examples

- Independent set: $\{\langle G, k \rangle : G$ has an independent set of size $k\}$. Recall the independent set is a set of vertices such that no two vertices are adjacent.
- Vertex cover: $\{\langle G, k \rangle : G$ has a subset of k vertices such that every edge has at least one end point in $S\}$
- Integer Linear Programming: $\{\langle A, b \rangle : Ax = b$ has a nontrivial solution $x \in \mathbb{Z}^n\}$
- SAT: $\{\langle F \rangle :$ Boolean formula such that there exists $x \in \{T/F\}^n$ and $F(x) = T\}$